# Chapter 17
# The bilinear covariant fields of the Dirac electron
# –
# from my book:
# Understanding Relativistic Quantum Field Theory

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# Contents

## Chapter 17

# The bilinear covariant fields of the Dirac electron

## 17.1 Bilinear fields of the 2d Dirac's equation

The spinor representation of the Dirac equations embodies the electro dynamic and kinematic properties of the electron like the charge/current density and the components of the spin density. All these values can be extracted using 4x4 gamma matrices in various combinations. The results are the so called *bilinear covariant fields*, The Lorentz scalar, pseudo scalar, vector and axial vector of the theory. We will these here first for the simpler case of the 2d Dirac equation.

All results given here are directly applicable for the full 4d dirac equation. The only limitations stem from the use of a single spatial dimension, for instance: The spin is always in the direction of the positive r-axis. The Dirac equation uses four 4x4 matrices corresponding with the four dimensions. Here we use two 2x2 $\Upsilon$-matrices corresponding with $t$ and $r$. The Dirac theory uses a 5th derived 4x4 matrix $\gamma^5$ for practical reasons. We will use an $\Upsilon^5$ matrix here defined in an equivalent way: $\Upsilon^5 = \Upsilon^t\Upsilon^r$. The following are

$$\Upsilon^t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Upsilon^r = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Upsilon^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{17.1}$$

$$\gamma^t = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \tag{17.2}$$

We see that the correspondence is complete, keeping in mind that $\sigma^t = I$ and $\sigma^5 = I$. All we need to do to from the 2d Dirac equation to the full 4d Dirac equation is to replace the "1"s with the appropriate Pauli spin matrices.

We now define the adjoint $\bar{\psi}$ of the wave-function $\psi$ with the help of the adjoint 'spinors' $\bar{u}$ and $\bar{v}$ (denoted by the bars) in exactly the same way as in the 4d Dirac theory, where $\bar{\psi}\psi$ plays a similar role as $\Psi^*\Psi$ in the non-relativistic theory. For particle and anti-particle plane-wave solutions we have respectively.

$$\psi \;=\; u\, e^{-Et/\hbar + ipr}, \qquad \bar{\psi} \;=\; \bar{u}\, e^{-Et/\hbar + ipr} \tag{17.3}$$

$$\psi \;=\; v\, e^{+Et/\hbar + ipr}, \qquad \bar{\psi} \;=\; \bar{v}\, e^{+Et/\hbar + ipr} \tag{17.4}$$

Where $\bar{u}$ and $\bar{v}$ are defined by.

$$\bar{u} \;=\; \Upsilon^t\; u^\dagger \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_L^* \\ u_R^* \end{pmatrix} \;=\; \begin{pmatrix} u_R^* \\ u_L^* \end{pmatrix} \tag{17.5}$$

$$\bar{v} \;=\; \Upsilon^t\; v^\dagger \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_L^* \\ v_R^* \end{pmatrix} \;=\; \begin{pmatrix} v_R^* \\ v_L^* \end{pmatrix} \tag{17.6}$$

## Generic Lorentz transformations

The advantage of first using a simpler equation is that it is easy to derive all the bilinear expressions. We first derive the behavior of the scalars and vector components under a boost.

$$\begin{pmatrix} u_L \\ u_R \end{pmatrix} \quad \overset{\text{boost}}{\Rightarrow} \quad \begin{pmatrix} \exp(-\vartheta/2)\;\; u_L \\ \exp(+\vartheta/2)\;\; u_R \end{pmatrix} \tag{17.7}$$

From this Lorentz transformed (boosted) spinor we can construct the combinations containing terms like $\psi_L^*\psi_L$. Since the exponents drops out the product here (=1) we can work with terms like $u_L^* u_L$ instead.

$$\begin{array}{lclcl} u_R^*u_L + u_L^*u_R & = & 2m & = & 2m \\ u_R^*u_L - u_L^*u_R & = & 0 & = & 0 \\ u_R^*u_R + u_L^*u_L & = & 2m\;\cosh\vartheta & = & 2m\;\gamma \\ u_R^*u_R - u_L^*u_L & = & 2m\;\sinh\vartheta & = & 2m\;\beta\gamma \end{array} \tag{17.8}$$

We can express these as bilinear combinations with the help of the adjoint spinors and the $\Upsilon$-matrices.

generic transformations

$$\begin{array}{lcll} \bar{u}\;u & = & (2m) & 1 \\ \bar{u}\;\Upsilon^t\;u & = & (2m) & \gamma \\ \bar{u}\;\Upsilon^r\;u & = & (2m) & \beta\gamma \end{array} \qquad \begin{array}{lcll} \bar{u}\;\Upsilon^5\;u & = & (2m) & 0 \\ \bar{u}\;\Upsilon^t\Upsilon^5\;u & = & (2m) & \beta\gamma \\ \bar{u}\;\Upsilon^r\Upsilon^5\;u & = & (2m) & \gamma \end{array} \tag{17.9}$$

We see that $\bar{u}u$ is a Lorentz scalar independent of the reference frame which can be associated with the mass of the particle. The bottom two rows can be combined into 2-vectors, $J_V$ the *vector* current and $J_A$ the *axial* current.

$$J_V = \bar{u}\, \Upsilon^\mu\, u = 2m(\ \gamma,\ \beta\gamma\ ) \tag{17.10}$$

$$J_A = \bar{u}\, \Upsilon^\mu \Upsilon^5\, u = 2m(\ \beta\gamma,\ \gamma\ ) \tag{17.11}$$

$J_V$ transforms as an energy/momentum vector while $J_A$ transforms as a spin vector. Currents associated with the left chiral component $\psi_L$ and the right chiral component $\psi_R$ can be expressed like.

$$J_L = u_L^* u_L = \frac{1}{2}\, \bar{u}\, \Upsilon^\mu \left(1 - \Upsilon^5\right)\, u \tag{17.12}$$

$$J_R = u_R^* u_R = \frac{1}{2}\, \bar{u}\, \Upsilon^\mu \left(1 + \Upsilon^5\right)\, u \tag{17.13}$$

The generic transformations can thus be mapped on the contravariant vectors for momentum $(E, p)$ and spin $s$ since these transform in the same way. We will see that this holds for four-vectors as well in the case of the Dirac equation.

$$\begin{pmatrix} u_L \\ u_R \end{pmatrix} \overset{\text{boost}}{\Rightarrow} \begin{pmatrix} \sqrt{(E - p^r)} \\ \sqrt{(E + p^r)} \end{pmatrix} \tag{17.14}$$

——— energy-momentum and spin ———

$$\begin{array}{llll} \bar{u}\, u = 2\ m & & \bar{u}\, \Upsilon^5\, u = (4m/\hbar)\ 0 \\ \bar{u}\, \Upsilon^t\, u = 2\ E & & \bar{u}\, \Upsilon^t \Upsilon^5\, u = (4m/\hbar)\ s^t \\ \bar{u}\, \Upsilon^r\, u = 2\ p^r & & \bar{u}\, \Upsilon^r \Upsilon^5\, u = (4m/\hbar)\ s^r \end{array} \tag{17.15}$$

Alternatively, we can map the charge/current density $J$ or magnetic momentum spin density $\mu$ on the bilinear expressions.

— charge-current density, magnetic moment density —

$$\begin{array}{llll} \bar{u}\, u = (2m/q)\ q & & \bar{u}\, \Upsilon^5\, u = 2m/(g_e \mu_B)\ 0 \\ \bar{u}\, \Upsilon^t\, u = (2m/q)\ j^t & & \bar{u}\, \Upsilon^t \Upsilon^5\, u = 2m/(g_e \mu_B)\ \mu^t \\ \bar{u}\, \Upsilon^r\, u = (2m/q)\ j^r & & \bar{u}\, \Upsilon^r \Upsilon^5\, u = 2m/(g_e \mu_B)\ \mu^r \end{array} \tag{17.16}$$

## 17.2 Bilinear fields of the 4d Dirac's equation

We can now go from the 2d bilinear covariant fields to the 4d bilinear covariant fields. The spin is introduced via the Pauli spin matrices $\sigma^i$ when we replace the 2x2 $\Upsilon$-matrices with the 4x4 $\gamma$-matrices:

$$\Upsilon^t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Upsilon^r = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Upsilon^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{17.17}$$

$$\gamma^t = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \tag{17.18}$$

The bilinear covariants of the 4d Dirac equation are listed below.

| **Bilinear expression** | | **transforms like a**: |
|---|---|---|
| $\bar{\psi}\psi$ | $1\times$ | scalar |
| $\bar{\psi}\gamma^\mu\psi$ | $4\times$ | vector |
| $\bar{\psi}\sigma^{\mu\nu}\psi$ | $6\times$ | antisymmetric tensor |
| $\bar{\psi}\gamma^\mu\gamma^5\psi$ | $4\times$ | axial vector |
| $\bar{\psi}\gamma^5\psi$ | $1\times$ | pseudoscalar |

(17.19)

Where the relation between $\bar{\psi}$ and $\psi$ is given by

$$\bar{\psi} \;=\; \gamma^t\, \psi^\dagger \;=\; \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \psi_L^\dagger \\ \psi_R^\dagger \end{pmatrix} \;=\; \begin{pmatrix} \psi_R^\dagger \\ \psi_L^\dagger \end{pmatrix} \tag{17.20}$$

With $\dagger$ denotes the complex conjugate. Next to the four we already knew from the 2d Dirac equation, the two Lorentz scalars and the two Lorentz vectors, we see a fifth form that represents a quantity which transforms as a tensor. This quantity is, not surprisingly, associated with the spin and occurs in spin interactions.

$$\sigma^{\mu\nu} \;=\; \frac{i}{2}[\gamma^\mu, \gamma^\nu] \;=\; \frac{i}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) \tag{17.21}$$

We see from (17.19) that there are in total 16 different 4x4 matrices which can be sandwiched in the bilinear expression, including $I$ the unity matrix. With these 16 matrices one can construct any 4x4 matrix as a linear combination.

In this section we'll have a general discussion about the physical meaning of the Dirac bilinears. The relevant insight-full derivations will then follow in the subsequent sections.

## The Lorentz scalar and pseudo scalar

The Lorentz scalar and pseudo scalar are the same in any reference frame under general Lorentz transform (boosts and rotations). The scalar $\bar{\psi}\psi = 2m$ represents the mass of the particle. The relation with the pseudo scalar becomes clearer when we show the explicit dependence on the left and right chiral components $\psi_L$ and $\psi_R$.

| Bilinear expression | LR decomposition | transforms like a: |
|---|---|---|
| $\bar{\psi}\psi$ | $\psi_L^\dagger\psi_R \; + \; \psi_R^\dagger\psi_L$ | scalar |
| $\bar{\psi}\gamma^5\psi$ | $\psi_L^\dagger\psi_R \; - \; \psi_R^\dagger\psi_L$ | pseudo scalar |

(17.22)

A pseudo scalar changes sign under parity inversion, the sign reversal of spacial axis. The quantity $\bar{\psi}\gamma^5\psi$ is always zero in the 2d theory. We will show in sections further on that the same hold for the 4d theory under very general conditions.

The condition that $\bar{\psi}\gamma^5\psi = 0$ has important consequences. Dirac showed in his first paper on his equation in 1928 that the orbital angular momentum and the spin angular momentum are only separately conserved if $\bar{\psi}\gamma^5\psi$ is zero.

The inversion of parity separately from time inversion depends on the reverence frame if the quantity does not propagate on the light-cone. The only value which remains the same under sign change is the value 0.

### The Lorentz vector and axial vector

The Lorentz vector and axial vector transform like charge/current density and axial current density respectively. The decomposition in $\psi_L$ and $\psi_R$ shows that both can be expressed with two independent chiral terms which describe the chiral currents $J_R$ and $J_L$.

| Bilinear expression | LR decomposition | transforms like a: |
|---|---|---|
| $j_V \;=\; \bar{\psi}\gamma^\mu\psi$ | $\psi_R^\dagger\sigma^\mu\psi_R \;+\; \psi_L^\dagger\tilde{\sigma}^\mu\psi_L$ | vector |
| $j_A \;=\; \bar{\psi}\gamma^\mu\gamma^5\psi$ | $\psi_R^\dagger\sigma^\mu\psi_R \;-\; \psi_L^\dagger\tilde{\sigma}^\mu\psi_L$ | axial vector |

(17.23)

So we can simply write $j_V = j_R + j_L$ and $j_A = j_R - j_L$. We have made use of a notation above involving $\sigma^\mu$ and $\tilde{\sigma}^\mu$ which are defined by.

$$\sigma^\mu = \left(\; \sigma^t + \sigma^x + \sigma^y + \sigma^z \right), \qquad \tilde{\sigma}^\mu = \left(\; \sigma^t - \sigma^x - \sigma^y - \sigma^z \right) \tag{17.24}$$

Where $\tilde{\sigma}^\mu$ is typically related to $\psi_L$ while $\sigma^\mu$ pairs with $\psi_R$. This notation also allows us to write the 4d Dirac equation in a more compact form. For example.

$$i \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix} \partial_\mu \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = m \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{17.25}$$

The vector quantity $j_V$ transforms like a contra-variant four vector. The time component transforms as $\gamma$ (like energy) and the spatial components transform like $\beta\gamma$ (like momentum). The time component transform like charge-*density*. The total charge is a Lorentz scalar, it is the same in all reference frames. The charge density however transforms like $\gamma$ since the total volume becomes Lorentz contracted with $\gamma$. The spatial component transforms like electric current-*density*. The total current depends only on the speed $\beta$ in any given reference frame. The current-density transforms like $\beta\gamma$ where the extra factor $\gamma$ comes again from the Lorentz contraction of the total volume.

The axial vector quantity $j_V$ transforms like an (inertial) spin-four-vector. The time component is zero in the rest-frame of the spin and transforms like $-\beta\gamma$. The spatial inertial spin-components transform like $\gamma$ along the direction of the boost while the spin-components orthogonal to the boost do not transform.

The time component of $j_A$ transforms like axial (electric) current density. The total axial current is Lorentz invariant

$$j_V = j_R + j_L, \qquad j_A = j_R - j_L \tag{17.26}$$

## The Lorentz (anti-symmetric) tensor

$$-\,\bar{\psi}\sigma^{\mu\nu}\psi \quad = \quad \begin{bmatrix} 0 & -P_x & -P_y & -P_z \\ P_x & 0 & -\tilde{M}_z & \tilde{M}_y \\ P_y & \tilde{M}_z & 0 & -\tilde{M}_x \\ P_z & -\tilde{M}_y & \tilde{M}_x & 0 \end{bmatrix} \tag{17.27}$$

| Bilinear expression | LR decomposition | transforms like the: |
|---|---|---|
| $\tilde{M}^k \;=\; \bar{\psi}\sigma^{ij}\psi$ | $\psi_L^\dagger\sigma^k\psi_R \;+\; \psi_R^\dagger\sigma^k\psi_L$ | Magnetization field |
| $P^j \;=\; \bar{\psi}\sigma^{0j}\psi$ | $i\psi_L^\dagger\sigma^j\psi_R \;-\; i\psi_R^\dagger\sigma^j\psi_L$ | Polarization field |

(17.28)

$$\partial_\nu F^{\mu\nu} \;=\; \mu_o j^\nu \tag{17.29}$$

$$\partial_\nu \left(\, \bar{\psi}\sigma^{\mu\nu}\psi \,\right) \;=\; \frac{2m}{e} j^\nu \tag{17.30}$$

## 17.3 Lorentz transform of spin four vectors

We will do a quick review here of the Lorentz transform of the classical axial current density vector to remind us what we should expect from the boost operations on spinors. Dirac's theory should give us the same results. The relativistic spin four-vector is equal to the classical three component vector in the rest-frame with the time-component set to zero

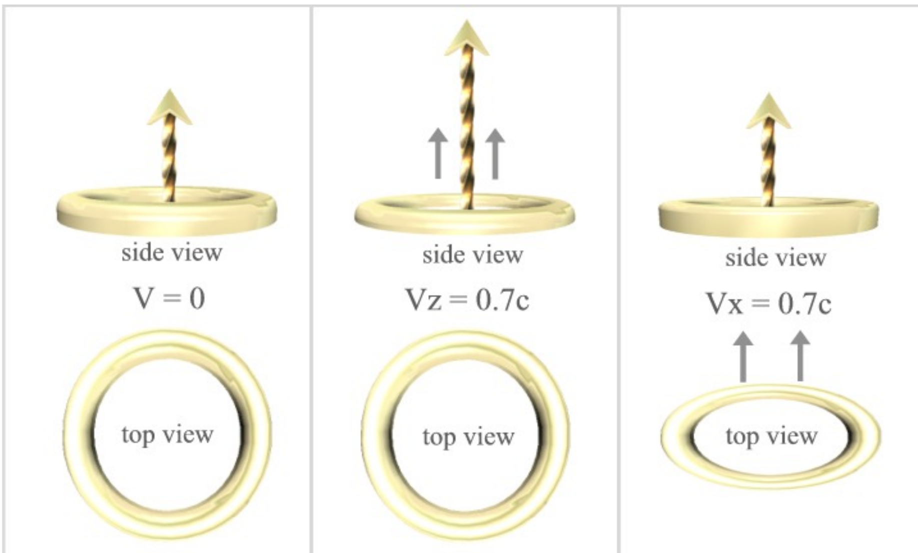


Figure 17.1: Lorentz transform of the spin four vector

Image 17.1 shows a circular current in its rest-frame (left), boosted parallel to the spin-axis (middle) and boosted orthogonal to the spin-axis (right). Recall that the magnetic moment is given by.

$$\mu \; = \; IA \qquad (17.31)$$

The total circular current is Lorentz invariant just as the total charge. The current times the area determines the (total) magnetic moment. Nothing changes if the boost is parallel to the rest spin. However, the area $A$ decreases with $\gamma$ in case of a boost orthogonal to the spin (left image). So, the magnetic moment $\mu$ is decreases in the orthogonal case while it stays unchanged for the parallel boost.

$$\vec{\mu}\,' \;=\; \frac{1}{\gamma}\,\vec{\mu}_\perp \;+\; \vec{\mu}_\| \tag{17.32}$$

The axial current density transforms with an extra factor $\gamma$ due to Lorentz contraction. The axial current density is constant in the orthogonal case while it increases with $\gamma$ for the parallel boost. The result is that for ultra-relativistic speeds the spin is always aligned with the velocity (except for the case where the spin is exactly orthogonal to the velocity)

$$\vec{j}_A\,' \;=\; \vec{j}_{A\perp} \;+\; \gamma\,\vec{j}_{A\|} \tag{17.33}$$

The inertial spin $\vec{s}$ transforms in the same way as the axial current density, because energy/momentum transforms with an extra factor $\gamma$ compared to the total charge/current. The time components of the four-vectors $\vec{\mu}$, $\vec{J}_A$ and $\vec{s}$ are per definition zero in the rest-frame. In other frames they are defined by the general law for the invariant of vector transformation. (Which implies that the constants are $-|\vec{\mu}|^2$, $-|\vec{J}_A|^2$ and $-|\vec{s}|^2$).

$$\mu_t^2/c^2 \;-\; \mu_x^2 - \mu_y^2 - \mu_z^2 \;=\; \text{constant} \tag{17.34}$$

## Magnetic moment density versus charge density

Note that an attempt to relate the circular current, which causes the magnetic moment in case of the (point) spin density, directly with the charge density would run into problems. The charge would have to move at speeds $>> c$ when the radius decreases to below the Compton radius to zero in case of a (theoretical) *point* spin.

One way out is to realize that the spin current can be both a positive charge density rotating counter-clockwise as well as a negative charge density rotating clockwise. So, one could for instance consider the magnetic moment to be augmented by a magnetic vacuum polarization in this way.

This problem doesn't occur in case of the inertial spin since the inertial momentum increases to $\infty$ when approaching c. There is no minimum radius in case of the inertial spin.

## 17.4 Lorentz transform of the vector/axial fields

We will first look into two elementary cases: A particle with spin z-up getting a boost in the x-direction and in the z-direction, orthogonal and parallel to spin. Following this we will derive a general formula.

### Boost in the x-direction orthogonal to the spin

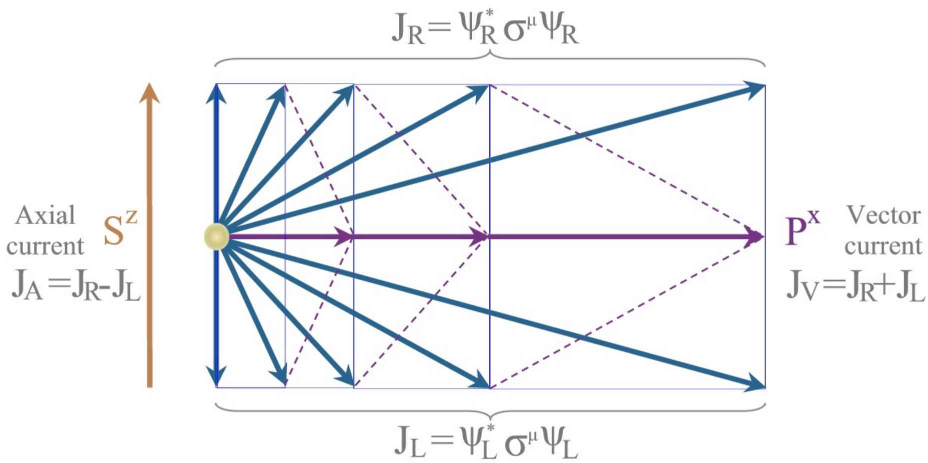


Figure 17.2: Transform of the chiral components under an x-boost

The spinor representing a spin up in the z-direction is $(1,\ 0)$. From equation (11.94) we see that it transforms into $(\ \cosh(\vartheta/2),\ \sinh(\vartheta/2\ )$ under a boost $\vartheta$ in the $x$-direction. We write out the components of $J_R = u_R^* \sigma^\mu u_R$

$$u_R^* \sigma^t u_R \;=\; \begin{pmatrix} \cosh\vartheta/2 \\ \sinh\vartheta/2 \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\vartheta/2 \\ \sinh\vartheta/2 \end{pmatrix} = \gamma \qquad (17.35)$$

$$u_R^* \sigma^x u_R \;=\; \begin{pmatrix} \cosh\vartheta/2 \\ \sinh\vartheta/2 \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cosh\vartheta/2 \\ \sinh\vartheta/2 \end{pmatrix} = \beta\gamma \qquad (17.36)$$

$$u_R^* \sigma^y u_R \;=\; \begin{pmatrix} \cosh\vartheta/2 \\ \sinh\vartheta/2 \end{pmatrix}^* \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \cosh\vartheta/2 \\ \sinh\vartheta/2 \end{pmatrix} = 0 \qquad (17.37)$$

$$u_R^* \sigma^z u_R \;=\; \begin{pmatrix} \cosh\vartheta/2 \\ \sinh\vartheta/2 \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cosh\vartheta/2 \\ \sinh\vartheta/2 \end{pmatrix} = 1 \qquad (17.38)$$

These are the four components of the current $j_R = u_R^* \sigma^\mu u_R$ which is associated with the right chiral component $\psi_R$. We have omitted the factor m, and used $\gamma = \cosh\vartheta$ and $\beta\gamma = \sinh\vartheta$. We are going to repeat this now for $j_L = u_L^* \tilde{\sigma}^\mu u_L \;=\; u_L^* (\; \sigma^t, \; -\sigma^x, \; -\sigma^y, \; -\sigma^z \;) u_L$, so that we can construct the vector current $j_V = j_R + j_L$ and the axial current $j_A = j_R - j_L$

The results are:

$$\begin{array}{lcr} j_R &=& m(\;\; \gamma \;\;\; \beta\gamma \;\;\; 0 \;\;\;\;\; 1 \;\;) \\ j_L &=& m(\;\; \gamma \;\;\; \beta\gamma \;\;\; 0 \;\; -1 \;\;) \\ j_V &=& 2m(\;\; \gamma \;\;\; \beta\gamma \;\;\; 0 \;\;\;\;\; 0 \;\;) \\ j_A &=& 2m(\;\; 0 \;\;\;\; 0 \;\;\;\; 0 \;\;\;\;\; 1 \;\;) \end{array} \tag{17.39}$$

The four currents are drawn in figure 17.2. We see that the vector current $j_V$ can be written as $2(E, p_x, 0, 0)$ as it should if it has to present the energy/momentum current. The axial current has a component 1 in the z-direction. Indeed, a boost orthogonal to the spin should not change the spin. $j_A$ is the same as it is in the rest-frame.

Next we check the results for this elementary case with the expressions for time-like, spin-like and light-like transformations as we require for the various currents.

**— Lorentz transforms of the vector, axial and chiral currents —**

$$\begin{array}{llcl} \text{Momentum:} & j_{Vt}^2 - j_{Vx}^2 - j_{Vy}^2 - j_{Vz}^2 &=& (2m)^2 \\ \text{Spin (axial):} & j_{At}^2 - j_{Ax}^2 - j_{Ay}^2 - j_{Az}^2 &=& -(2m)^2 \\ \text{Left chiral:} & j_{Lt}^2 - j_{Lx}^2 - j_{Ly}^2 - j_{Lz}^2 &=& 0 \\ \text{Right chiral:} & j_{Rt}^2 - j_{Rx}^2 - j_{Ry}^2 - j_{Rz}^2 &=& 0 \end{array} \tag{17.40}$$

By inserting the results of (17.39) into table (17.40) we can check that our elementary example indeed behaves as we require. The fact that $j_R$ and $j_L$ transform light-like implies that they propagate with the speed of light just as in the case of the 2d Dirac equation.

### Boost in the z-direction parallel to the spin

The z-up spinor (1, 0) becomes ( exp($\vartheta$/2), 0 ) as a result from a boost in the $z$-direction parallel to the spin, see equation (11.99) Writing out the four components of the current $J_R = u_R^* \sigma^\mu u_R$ gives us:

$$u_R^* \sigma^t u_R = \begin{pmatrix} \exp\frac{\vartheta}{2} \\ 0 \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \exp\frac{\vartheta}{2} \\ 0 \end{pmatrix} = \exp\vartheta \qquad (17.41)$$

$$u_R^* \sigma^x u_R = \begin{pmatrix} \exp\frac{\vartheta}{2} \\ 0 \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \exp\frac{\vartheta}{2} \\ 0 \end{pmatrix} = 0 \qquad (17.42)$$

$$u_R^* \sigma^y u_R = \begin{pmatrix} \exp\frac{\vartheta}{2} \\ 0 \end{pmatrix}^* \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \exp\frac{\vartheta}{2} \\ 0 \end{pmatrix} = 0 \qquad (17.43)$$

$$u_R^* \sigma^z u_R = \begin{pmatrix} \exp\frac{\vartheta}{2} \\ 0 \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \exp\frac{\vartheta}{2} \\ 0 \end{pmatrix} = \exp\vartheta \qquad (17.44)$$

Repeating this for $j_L = u_L^* \tilde{\sigma}^\mu u_L$ allows us to calculate the vector and axial currents.

$$\begin{aligned} j_R &= m( \exp\vartheta \quad 0 \quad 0 \quad +\exp\vartheta \;) \\ j_L &= m( \exp\vartheta \quad 0 \quad 0 \quad -\exp\vartheta \;) \\ j_V &= 2m( \gamma \quad 0 \quad 0 \quad \beta\gamma \;) \\ j_A &= 2m( \beta\gamma \quad 0 \quad 0 \quad \gamma \;) \end{aligned} \qquad (17.45)$$

The vector current $j_V$ can be written as $2(E, 0, 0, p_z)$ as it should be. The axial current $j_A$ has a time component which is zero in the rest-frame. The z-component of the axial current (spin) has grown from 1 to $\gamma$ as expected for a boost parallel to the spin.

This elementary case (boost parallel to the spin) is identical to the 2d Dirac equation case, as we can see by comparing (17.45) with (17.9)

Inserting (17.45) into the table 17.40 confirms that $j_R$ and $j_L$ transform light-like. $j_V$ transforms as an energy/momentum vector (as well as a charge/current density vector), and $j_A$ transforms as a spin.

## The derivation of the general Lorentz transformation

For the general boost formula we'll take an arbitrary spinor $\xi$ in its rest-frame and give it a boost in an arbitrary direction. For a plane-wave solution the left and right 2-spinors are equal in the rest frame. After a boost they become generally different.

$$\psi \;=\; \begin{pmatrix} \xi \\ \xi \end{pmatrix} e^{-iE_o t/\hbar} \quad \overset{\text{boost}}{\Rightarrow} \quad \begin{pmatrix} \xi'_L \\ \xi'_R \end{pmatrix} e^{-iEt/\hbar + i\vec{p}\cdot\vec{x}/\hbar} \tag{17.46}$$

We recall equation (11.91) here for the general boost operator.

$$\begin{pmatrix} \psi_L \\ \\ \psi_R \end{pmatrix} \quad \overset{\text{boost}}{\Rightarrow} \quad \begin{pmatrix} \exp\left(-\ \vartheta/2\ \ \sigma^i\right)\ \psi_L \\ \\ \exp\left(+\ \vartheta/2\ \ \sigma^i\right)\ \psi_R \end{pmatrix} \tag{17.47}$$

Where $i$ denotes the $x$, $y$ or $z$-direction. For the right and left chiral components we can write the 2-spinor boost transform as. $(R = +, L = -)$

$$\xi'_{R/L} \quad \overset{\text{boost}}{=} \quad \exp\left(\pm\frac{\vartheta}{2}\ \sigma^i\right)\xi \;=\; I\cosh\left(\frac{\vartheta}{2}\right)\xi \;\pm\; \sigma^i \sinh\left(\frac{\vartheta}{2}\right)\xi \tag{17.48}$$

The currents corresponding with the right and left chiral components are.

$$j^\mu_R \;=\; \xi'^*_R\ \sigma^\mu\ \xi'_R \;=\; \xi'^*_R\ (\ \sigma^t,\ +\sigma^x,\ +\sigma^y,\ +\sigma^z\ )\ \xi'_R \tag{17.49}$$

$$j^\mu_L \;=\; \xi'^*_L\ \tilde{\sigma}^\mu\ \xi'_L \;=\; \xi'^*_L\ (\ \sigma^t,\ -\sigma^x,\ -\sigma^y,\ -\sigma^z\ )\ \xi'_L \tag{17.50}$$

Working this out for $j^\mu_R$ gives us.

$$\begin{aligned} j^\mu_R \;=\; & \left(\xi^*\sigma^\mu\xi\right)\ \cosh^2\left(\frac{\vartheta}{2}\right) \;+\; \left(\xi^*(\sigma^i\sigma^\mu\sigma^i)\xi\right)\ \sinh^2\left(\frac{\vartheta}{2}\right) \;+ \\ & +\;\left(\xi^*(\sigma^i\sigma^\mu + \sigma^\mu\sigma^i)\xi\right)\ \sinh\left(\frac{\vartheta}{2}\right)\cosh\left(\frac{\vartheta}{2}\right) \end{aligned} \tag{17.51}$$

Note that we have removed the transformed $\xi'_R$ here. The only spinors in the formula above are the rest-frame $\xi$ here. We have used the important identity.

$$\left(\sigma^\beta\ \xi\right)^* \cdot \left(\sigma^\lambda\ \xi\right) \;=\; \xi^*\left(\sigma^\beta\ \sigma^\lambda\right)\xi \tag{17.52}$$

Since products of pauli sigma matrices are also sigma matrices we will obtain an expression containing only terms like $\xi^*\sigma^i\xi$ which represent the projection of the spin pointer on the $i$-axis. This will make it easy to interpret the final results.

Next we want to remove the $\vartheta/2$ arguments and replace them with hyperbolic functions which carry $\vartheta$ itself as argument since we can readily interpret these with the relativistic transformation factors $\gamma$ and $\beta$.

$$
\begin{aligned}
j_R^\mu \;=\; & \frac{1}{2}\Big(\xi^*(\sigma^\mu + \sigma^i\sigma^\mu\sigma^i)\xi\Big)\;\cosh(\vartheta) \;+\; \frac{1}{2}\Big(\xi^*(\sigma^\mu - \sigma^i\sigma^\mu\sigma^i)\xi\Big)\;+ \\
& + \frac{1}{2}\Big(\xi^*(\sigma^i\sigma^\mu + \sigma^\mu\sigma^i)\xi\Big)\;\sinh(\vartheta)
\end{aligned}
\qquad (17.53)
$$

It's advantageous to separate the time component from the spatial components here because $\sigma^i\sigma^0 = \sigma^0\sigma^i$ while $\sigma^i\sigma^j = -\sigma^j\sigma^i$. The terms in the above expression will either cancel or otherwise simplify depending on $\mu$ being either 0 or $1, 2, 3$.

We replace the hyperbolic functions cosh and sinh with $\gamma$ and $\beta\gamma$. The Lorentz scalars $\xi^*\xi$ are replaced by $m$ and the projections like $\xi^*\sigma^i\xi$ are replaced by explicit dot-products as $m(\hat{s}\cdot\hat{x}^i)$, where $\hat{s}$ is the unit vector in the direction of the spin. This all now leads us to.

$$
\begin{aligned}
& j_R^0 = m\,(\;\gamma \;+\; (\hat{s}\cdot\hat{x}^i)\;\beta\gamma\;) && j_R^j = m\,\Big(\;\delta^{ij}\beta\gamma + \overline{\delta^{ij}}\;\hat{s}\cdot\hat{x}^j + \delta^{ij}\;\hat{s}\cdot\hat{x}^i\;\gamma\;\Big) \\
& j_L^0 = m\,(\;\gamma \;-\; (\hat{s}\cdot\hat{x}^i)\;\beta\gamma\;) && j_L^j = m\,\Big(\;\delta^{ij}\beta\gamma - \overline{\delta^{ij}}\;\hat{s}\cdot\hat{x}^j - \delta^{ij}\;\hat{s}\cdot\hat{x}^i\;\gamma\;\Big) \\
& j_V^0 = 2m\;\gamma && j_V^j = 2m\;\delta^{ij}\beta\gamma \\
& j_A^0 = 2m\;(\hat{s}\cdot\hat{x}^i)\;\beta\gamma && j_A^j = 2m\;(\;\overline{\delta^{ij}}\;\hat{s}\cdot\hat{x}^j + \delta^{ij}\;\hat{s}\cdot\hat{x}^i\;\gamma\;)
\end{aligned}
\qquad (17.54)
$$

Where $\overline{\delta^{ij}}$ is the opposite of $\delta^{ij}$, it is 1 when $i \neq j$. With the help of the above equations (17.54) we can check the behavior of these currents under an arbitrary boost.

With respect to the vector current $j_V$ we find for the energy $j_V^0 = 2E$, and for the momentum $j_V^j = \delta^{ij}\ 2p_j$, the momentum is in the direction of the boost due to the $\delta^{ij}$.

The axial current $j_V$ contains only spin-projections like $(\hat{s}\cdot\hat{x}^i)$. For the spatial of the four-vector we recall equation (17.33) for the transform of the magnetic moment: $\vec{\mu}' = \vec{\mu}_\perp + \gamma\ \vec{\mu}_{||}$. The boost only affects the spin component parallel to the boost which increases by a factor $\gamma$. We see this back in $j_A^j$ where the spin component parallel to the boost is $\hat{s}\cdot\hat{x}^i\gamma$ and the components orthogonal to the boost are $\hat{s}\cdot\hat{x}^j$.

The expressions (17.54) contain terms like $\overline{\delta^{ij}}$ and $\delta^{ij}$ because we started with the Pauli matrices of the principle axis $\sigma^i$ expressing boosts in the direction of the principle axis. We can generalize this to arbitrary boost in vector algebra.

**Lorentz transforms of the Vector and Axial currents**

$$
\begin{array}{ll}
j_R^0 = m\left(\ 1 + \hat{s}\cdot\vec{\beta}\ \right)\gamma & \vec{j}_R = m\left(\ \vec{\beta}\gamma + \hat{s}_\perp + \hat{s}_{||}\gamma\ \right) \\
j_L^0 = m\left(\ 1 - \hat{s}\cdot\vec{\beta}\ \right)\gamma & \vec{j}_L = m\left(\ \vec{\beta}\gamma - \hat{s}_\perp - \hat{s}_{||}\gamma\ \right) \\
j_V^0 = 2m\ \gamma & \vec{j}_V = 2m\ \vec{\beta}\gamma \\
j_A^0 = 2m\ (\hat{s}\cdot\vec{\beta})\ \gamma & \vec{j}_A = 2m\ (\ \hat{s}_\perp + \hat{s}_{||}\gamma\ )
\end{array}
\qquad (17.55)
$$

Where $\hat{s}_{||} = (\hat{\beta}\cdot\hat{s})\ \hat{\beta}$ and $\hat{s}_\perp = \hat{s} - \hat{s}_{||}$ are the parallel and orthogonal components of the spin unit vector $\hat{s}$ in the rest frame with respect to $\vec{\beta}$. The general expressions of (17.55) confirm that the quantum mechanical Dirac vector current and axial current transform in exactly the same way as their classical counter parts.

We will check the expressions against the table 17.40. The general expressions for the left-chiral $j_L$ and right-chiral $j_R$ current is light-like which means that they propagate with c. The condition for $j_R$ to transform lightlike, like a massless component is given by:

$$j_{Rt}^2 \;-\; j_{Rx}^2 \;-\; j_{Ry}^2 \;-\; j_{Rz}^2 \;=\; 0 \tag{17.56}$$

If we apply this to our results then we get for the time component and the spatial component repectively.

$$(j_{Rt})^2 \;=\; m^2\Big(\gamma \;+\; |\hat{s}_{\|}|\;\beta\gamma\;\Big)^2 \tag{17.57}$$

$$(j_R^j)^2 \;=\; m^2\Big(\vec{\beta}+\hat{s}_{\|}\Big)^2\gamma^2 \quad + \quad m^2\Big(\,\hat{s}_{\perp}\Big)^2 \tag{17.58}$$

Where the spatial component has two orthogonal components parallel and transversal to the direction of motion. Collecting all the terms in one expression gives us.

$$(j_{Rt})^2-(j_R^j)^2 \;=$$

$$m^2\Big((1-\beta^2)\gamma^2 \;+\; 2(\;\hat{s}_{\|}\cdot\vec{\beta}\;-\hat{s}_{\|}\cdot\vec{\beta}\;)\gamma \;-\; \hat{s}_{\|}^2(1-\beta^2)\gamma^2 \;-\; \hat{s}_{\perp}^2\;\Big) \tag{17.59}$$

With $(1-\beta^2)\gamma^2 = 1$ per definition this becomes.

$$(j_{Rt})^2-(j_R^j)^2 \;=\; m^2\Big(1 \;-\; \hat{s}_{\|}^2 \;-\; \hat{s}_{\perp}^2\;\Big) \;=\; 0 \tag{17.60}$$

Which is 0 since $|\hat{s}| = 1$. We have hereby proved that $j_R$ transforms light-like. The equivalent prove for $j_L$ just needs some sign changes. This completes our prove for the light like behavior of the chiral components.

In the specific case in which one zeroes out either the left chiral or the right chiral part of the four spinor, then the vector current and axial current become equal $j_A = j_V$ or reversed $j_A = -j_V$. The spin is in parallel or anti-parallel to the momentum, in all reference frames. This is only possible for light-like transforming particles. In this specific case this is easy to prove. We have shown here that this property actually holds in general.

### Reflection of the chiral current by spin

(to be finished)

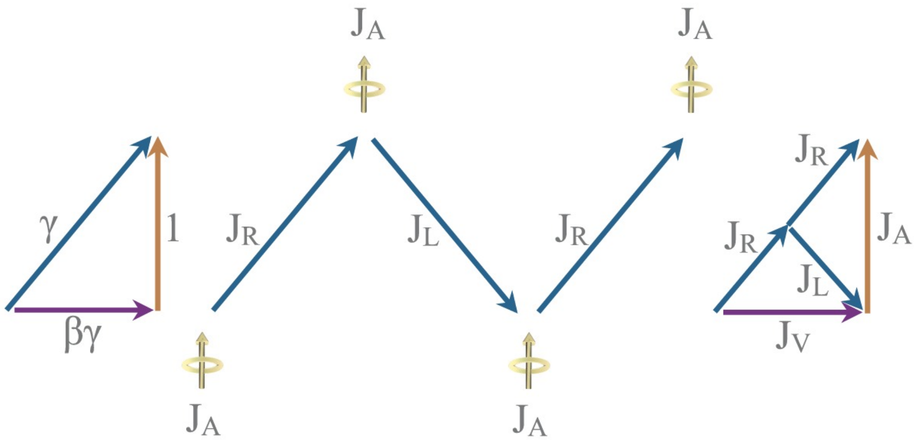


Figure 17.3: Reflection of the chiral current by the spin

## 17.5 Lorentz transform of the scalar fields

| Bilinear expression | LR decomposition | transforms like a: |
|---|---|---|
| $\bar{\psi}\,\psi$ | $\psi_L^*\,\psi_R \;+\; \psi_R^*\,\psi_L$ | scalar |
| $i\bar{\psi}\gamma^5\psi$ | $i\psi_L^*\,\psi_R \;-\; i\psi_R^*\,\psi_L$ | pseudo scalar |

(17.61)

$$\xi_R' \overset{\text{boost}}{=} \exp\left(+\frac{\vartheta}{2}\,\sigma^i\right)\xi, \qquad \xi_L' \overset{\text{boost}}{=} \exp\left(-\frac{\vartheta}{2}\,\sigma^i\right)\xi \tag{17.62}$$

Working this out for $J^j_{RL} = \psi_R^*\,\psi_L$ and $J^j_{LR} = \psi_L^*\,\psi_R$ gives us.

$$J^j_{RL} \;=\; \xi^*\xi\;, \qquad J^j_{LR} \;=\; \xi^*\xi \tag{17.63}$$

$$\bar{\psi}\,\psi \;=\; 2m, \qquad i\bar{\psi}\gamma^5\psi \;=\; 0 \tag{17.64}$$

## 17.6 Lorentz transform of the tensor field

The antisymmetric tensor $\bar{\psi}\sigma^{\mu\nu}\psi$ is the most complex of the Dirac bilinear quantities. It can be expressed in the form of $\gamma$ commutators.

$$\sigma^{\mu\nu} \;=\; \frac{i}{2}\,[\;\gamma^\mu,\gamma^\nu\;]\,, \qquad \text{with} \quad [\gamma^\mu,\gamma^\nu] = (\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) \tag{17.65}$$

The various 4x4 matrices are equal to the general Lorentz transformation generator $J^{\mu\nu}$ for boosts $K_i$ and rotations $L_i$

$$\mathrm{J}^{\mu\nu} \;=\; \frac{i}{2}\sigma^{\mu\nu} \;=\; \begin{bmatrix} 0 & -K_x & -K_y & -K_z \\ K_x & 0 & -L_z & L_y \\ K_y & L_z & 0 & -L_x \\ K_z & -L_y & L_x & 0 \end{bmatrix} \tag{17.66}$$

With all combinations written out in Pauli $\sigma$ matrices we get.

$$\sigma^{\mu\nu} \;=\;$$

$$\begin{bmatrix}
\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} \sigma^x & 0 \\ 0 & -\sigma^x \end{bmatrix} & \begin{bmatrix} \sigma^y & 0 \\ 0 & -\sigma^y \end{bmatrix} & \begin{bmatrix} \sigma^z & 0 \\ 0 & -\sigma^z \end{bmatrix} \\
\begin{bmatrix} -\sigma^x & 0 \\ 0 & \sigma^x \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} i\sigma^z & 0 \\ 0 & i\sigma^z \end{bmatrix} & \begin{bmatrix} -i\sigma^y & 0 \\ 0 & -i\sigma^y \end{bmatrix} \\
\begin{bmatrix} -\sigma^y & 0 \\ 0 & \sigma^y \end{bmatrix} & \begin{bmatrix} -i\sigma^z & 0 \\ 0 & -i\sigma^z \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} i\sigma^x & 0 \\ 0 & i\sigma^x \end{bmatrix} \\
\begin{bmatrix} -\sigma^z & 0 \\ 0 & \sigma^z \end{bmatrix} & \begin{bmatrix} i\sigma^y & 0 \\ 0 & i\sigma^y \end{bmatrix} & \begin{bmatrix} -i\sigma^x & 0 \\ 0 & -i\sigma^x \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\end{bmatrix} \tag{17.67}$$

We will show that the bilinear $\bar{\psi}\sigma^{\mu\nu}\psi$ can be associated with the magnetization/ polarization tensor which describes the electromagnetic fields due to the continuously distributed spin of the wave function.

$$-\bar{\psi}\sigma^{\mu\nu}\psi = \begin{bmatrix} 0 & -P_x & -P_y & -P_z \\ P_x & 0 & -M_z & M_y \\ P_y & M_z & 0 & -M_x \\ P_z & -M_y & M_x & 0 \end{bmatrix} \tag{17.68}$$

We see the relation between the magnetization and polarization vector if we decompose the vectors in products involving the left and right chiral components $\psi_L$ and $\psi_R$. We will show that the magnetization and polarization vectors transform like the magnetic and electric field respectively.

| Bilinear expression | LR decomposition | transforms like the: |
|---|---|---|
| $M^k = \bar{\psi}\sigma^{ij}\psi$ | $\psi_L^*\sigma^k\psi_R + \psi_R^*\sigma^k\psi_L$ | Magnetic field |
| $P^j = i\bar{\psi}\sigma^{0j}\psi$ | $i\psi_L^*\sigma^j\psi_R - i\psi_R^*\sigma^j\psi_L$ | Electric field |

(17.69)

If $\xi$ is the two component Pauli spinor representing the spin in the rest frame then we can define the transform of $\xi$ for the left and right chiral component as follows.

$$\xi'_R \overset{\text{boost}}{=} \exp\left(+\frac{\vartheta}{2}\,\sigma^i\right)\xi = I\cosh(\frac{\vartheta}{2})\xi + \sigma^i\sinh(\frac{\vartheta}{2})\xi \tag{17.70}$$

$$\xi'_L \overset{\text{boost}}{=} \exp\left(-\frac{\vartheta}{2}\,\sigma^i\right)\xi = I\cosh(\frac{\vartheta}{2})\xi - \sigma^i\sinh(\frac{\vartheta}{2})\xi \tag{17.71}$$

Working this out for $J^j_{RL} = \psi_R^*\sigma^j\psi_L$ and $J^j_{LR} = \psi_L^*\sigma^j\psi_R$ gives us.

$$\begin{aligned} J^j_{RL} &= \left(\xi^*\sigma^j\xi\right)\cosh^2\left(\frac{\vartheta}{2}\right) - \left(\xi^*(\sigma^i\sigma^j\sigma^i)\xi\right)\sinh^2\left(\frac{\vartheta}{2}\right) + \\ &\quad + \left(\xi^*(\sigma^i\sigma^j - \sigma^j\sigma^i)\xi\right)\sinh\left(\frac{\vartheta}{2}\right)\cosh\left(\frac{\vartheta}{2}\right) \end{aligned} \tag{17.72}$$

$$\begin{aligned} J^j_{LR} &= \left(\xi^*\sigma^j\xi\right)\cosh^2\left(\frac{\vartheta}{2}\right) - \left(\xi^*(\sigma^i\sigma^j\sigma^i)\xi\right)\sinh^2\left(\frac{\vartheta}{2}\right) + \\ &\quad - \left(\xi^*(\sigma^i\sigma^j - \sigma^j\sigma^i)\xi\right)\sinh\left(\frac{\vartheta}{2}\right)\cosh\left(\frac{\vartheta}{2}\right) \end{aligned} \tag{17.73}$$

Where we have eliminated the transformed $\xi'_R$ and $\xi'_L$ Pauli spinors here. The only Pauli spinor in the formulas above is the rest-frame $\xi$. We made use of the identity.

$$\left(\sigma^\beta\ \xi\right)^* \cdot \left(\sigma^\lambda\ \xi\right) \quad = \quad \xi^* \left(\sigma^\beta\ \sigma^\lambda\right) \xi \tag{17.74}$$

Next we want to remove the $\vartheta/2$ arguments and replace them with hyperbolic functions which carry $\vartheta$ itself as argument since we can readily interpret these with the relativistic transformation factors $\gamma$ and $\beta$.

$$\cosh^2(x) = \frac{1}{2}(\cosh(2x)+1), \qquad \sinh^2(x) = \frac{1}{2}(\cosh(2x)-1)$$

$$\sinh(x)\cosh(x) = \frac{1}{2}\sinh(2x) \tag{17.75}$$

With the above standard identities we get.

$$\begin{aligned} J^j_{RL} \ &= \ \frac{1}{2}\Big(\xi^*(\sigma^j - \sigma^i\sigma^j\sigma^i)\xi\Big)\ \cosh(\vartheta)\ +\ \frac{1}{2}\Big(\xi^*(\sigma^j + \sigma^i\sigma^j\sigma^i)\xi\Big)\ + \\ &\quad +\ \frac{1}{2}\Big(\xi^*(\sigma^i\sigma^j - \sigma^j\sigma^i)\xi\Big)\ \sinh(\vartheta) \end{aligned} \tag{17.76}$$

$$\begin{aligned} J^j_{LR} \ &= \ \frac{1}{2}\Big(\xi^*(\sigma^j - \sigma^i\sigma^j\sigma^i)\xi\Big)\ \cosh(\vartheta)\ +\ \frac{1}{2}\Big(\xi^*(\sigma^j + \sigma^i\sigma^j\sigma^i)\xi\Big)\ + \\ &\quad -\ \frac{1}{2}\Big(\xi^*(\sigma^i\sigma^j - \sigma^j\sigma^i)\xi\Big)\ \sinh(\vartheta) \end{aligned} \tag{17.77}$$

We replace the hyperbolic functions cosh and sinh with $\gamma$ and $\beta\gamma$. The Lorentz scalars $\xi^*\xi$ are replaced by $m$ and the projections like $\xi^*\sigma^i\xi$ are replaced by explicit dot-products as $m(\hat{s}\cdot\hat{x}^i)$, where $\hat{s}$ is the unit vector in the direction of the spin. This all now leads us to.

Products of pauli sigma matrices are also sigma matrices and we will obtain expressions containing only terms like $\xi^*\sigma^i\xi$ which represent the projection of the spin pointer on the $i$-axis in order to make it easier to interpret the final results.

$$
\begin{aligned}
J^j_{LR} &= \quad m\left(-\overline{\delta^{ij}}\ \hat{s}\cdot\hat{x}^k\ \beta\gamma i \ + \ \delta^{ij}\ \hat{s}\cdot\hat{x}^j \ + \ \overline{\delta^{ij}}\ \hat{s}\cdot\hat{x}^i\ \gamma\ \right) \\
J^j_{RL} &= \quad m\left(\ \overline{\delta^{ij}}\ \hat{s}\cdot\hat{x}^k\ \beta\gamma i \ + \ \delta^{ij}\ \hat{s}\cdot\hat{x}^j \ + \ \overline{\delta^{ij}}\ \hat{s}\cdot\hat{x}^i\ \gamma\ \right) \\
P^j &= -2m\left(\ \overline{\delta^{ij}}\ \hat{s}\cdot\hat{x}^k\ \beta\gamma i\ \right) \\
M^j &= \quad 2\,m\left(\ \delta^{ij}\ \hat{s}\cdot\hat{x}^j \ + \ \overline{\delta^{ij}}\ \hat{s}\cdot\hat{x}^i\ \gamma\ \right)
\end{aligned}
\tag{17.78}
$$

Where $\overline{\delta^{ij}}$ is the opposite of $\delta^{ij}$, it is 1 when $i \neq j$. The expressions (17.78) contain terms like $\overline{\delta^{ij}}$ and $\delta^{ij}$ because we started with the Pauli matrices of the principle axis $\sigma^i$ expressing boosts in the direction of the principle axis. We can generalize this to arbitrary boost in vector algebra.

### Lorentz transform of the anti-symmetric tensor components

$$
\boxed{
\begin{aligned}
\vec{J}_{LR} &= \quad m\left(\ \hat{s}_\otimes\ \beta\gamma i \ + \ \hat{s}_\parallel \ + \ \hat{s}_\perp\gamma\ \right) \\
\vec{J}_{RL} &= \quad m\left(-\hat{s}_\otimes\ \beta\gamma i \ + \ \hat{s}_\parallel \ + \ \hat{s}_\perp\gamma\ \right) \\
\vec{P} &= 2m\left(\ \hat{s}_\otimes\ \beta\gamma i\ \right) \\
\vec{M} &= 2m\left(\ \hat{s}_\parallel \ + \ \hat{s}_\perp\gamma\ \right)
\end{aligned}
}
\tag{17.79}
$$

We have made use of the shorthand notations $\hat{s}_\parallel$, $\hat{s}_\perp$ and $\hat{s}_\otimes$ for the components of the unit vector $\hat{s}$ of the spin in the particles rest frame with regard to the boost $\beta$. They are defined by.

$$
\begin{aligned}
\hat{s}_\parallel &= (\ \hat{\beta}\ \cdot\ \hat{s}\ )\ \hat{\beta} && \text{parallel component with regard to } \vec{\beta} \\
\hat{s}_\perp &= (\ \hat{\beta}\times\hat{s}\ )\times\hat{\beta} && \text{orthogonal component with regard to } \vec{\beta} \\
\hat{s}_\otimes &= (\ \hat{\beta}\times\hat{s}\ ) && 90^o \text{ rotated orthogonal component}
\end{aligned}
\tag{17.80}
$$

Where $\hat{s} = \hat{s}_\parallel + \hat{s}_\perp$ is always true.

The standard Lorentz transform of the EM-field under a boost in the x-direction is.

$$\begin{aligned} E'_x &= E_x & B'_x &= B_x \\ E'_y &= \gamma(E_y - \beta B_z) & B'_y &= \gamma(B_y + \beta E_z) \\ E'_z &= \gamma(E_z + \beta B_y) & B'_z &= \gamma(B_z - \beta E_y) \end{aligned} \tag{17.81}$$

Which we can rewrite for an arbitrary boost to a form which also uses our short hand notation for the various components.

**Lorentz transform of the electromagnetic field**

$$\begin{aligned} \mathsf{E}' &= \mathsf{E}_{\|} + \mathsf{E}_{\perp}\,\gamma + \mathsf{B}_{\otimes}\,\beta\gamma \\ \mathsf{B}' &= \mathsf{B}_{\|} + \mathsf{B}_{\perp}\,\gamma - \mathsf{E}_{\otimes}\,\beta\gamma \end{aligned} \tag{17.82}$$

Where we have made use of the following shorthand notations.

$$\begin{aligned} \mathsf{E}_{\|} &= (\,\hat{\beta}\cdot\mathsf{E}\,)\,\hat{\beta} && \text{parallel component with regard to } \vec{\beta} \\ \mathsf{E}_{\perp} &= (\,\hat{\beta}\times\mathsf{E}\,)\times\hat{\beta} && \text{orthogonal component with regard to } \vec{\beta} \\ \mathsf{E}_{\otimes} &= (\,\hat{\beta}\times\mathsf{E}\,) && 90^o \text{ rotated orthogonal component} \\ \\ \mathsf{B}_{\|} &= (\,\hat{\beta}\cdot\mathsf{B}\,)\,\hat{\beta} && \text{parallel component with regard to } \vec{\beta} \\ \mathsf{B}_{\perp} &= (\,\hat{\beta}\times\mathsf{B}\,)\times\hat{\beta} && \text{orthogonal component with regard to } \vec{\beta} \\ \mathsf{B}_{\otimes} &= (\,\hat{\beta}\times\mathsf{B}\,) && 90^o \text{ rotated orthogonal component} \end{aligned} \tag{17.83}$$

We can now compare the transformation of the magnetization and polarization vectors from the rest frame to a boosted frame, one-to-one with the general Lorentz transform of the electromagnetic field field.

We see that only the magnetization vector $\vec{M}$ is non-zero in the rest-frame where $\vec{M} = 2m\hat{s}$. The polarization vector $\vec{P}$ is zero in the rest frame. Under a boost $\vec{M}$ is transformed like the magnetic field B and the polarization $\vec{P}$ is a result from the boosted magnetization like the electric field E arises from the boosted magnetic field B.

## 17.7 Overview of the bilinear field transforms

Lorentz transforms of the bilinear Dirac fields from the (local) rest frame where $\hat{s}$ is the unit (rest) spin vector under a boost $\vec{\beta}$ with the assumption that the chiral components are equal: $\psi_R = \psi_L$, in the (local) rest-frame.

**Bilinear scalar- and pseudo scalar field transform**

$$\bar{\psi}\psi \;=\; 2m\,|\phi|^2 \qquad \bar{\psi}\gamma^5\psi \;=\; 0 \tag{17.84}$$

**Bilinear vector- and axial vector field transform**

$$\bar{\psi}\gamma^\mu\psi \;=\bar{\psi}\psi \begin{bmatrix} \gamma \\ \vec{\beta}_x\,\gamma \\ \vec{\beta}_y\,\gamma \\ \vec{\beta}_z\,\gamma \end{bmatrix} \qquad \bar{\psi}\gamma^\mu\gamma^5\psi \;=\bar{\psi}\psi \begin{bmatrix} \gamma(\hat{s}\cdot\vec{\beta}) \\ (\hat{s}_\perp + \hat{s}_\|\gamma)_x \\ (\hat{s}_\perp + \hat{s}_\|\gamma)_y \\ (\hat{s}_\perp + \hat{s}_\|\gamma)_z \end{bmatrix} \tag{17.85}$$

**Bilinear tensor field transform**

$$\bar{\psi}\sigma^{\mu\nu}\psi \;=$$

$$\frac{\bar{\psi}\psi}{2}\begin{bmatrix} 0 & -(\;\hat{s}_\otimes\,\beta\gamma\;)_x & -(\;\hat{s}_\otimes\,\beta\gamma\;)_y & -(\;\hat{s}_\otimes\,\beta\gamma\;)_z \\ (\;\hat{s}_\otimes\,\beta\gamma\;)_x & 0 & i(\hat{s}_\| + \hat{s}_\perp\gamma)_z & -i(\hat{s}_\| + \hat{s}_\perp\gamma)_y \\ (\;\hat{s}_\otimes\,\beta\gamma\;)_y & -i(\hat{s}_\| + \hat{s}_\perp\gamma)_z & 0 & i(\hat{s}_\| + \hat{s}_\perp\gamma)_x \\ (\;\hat{s}_\otimes\,\beta\gamma\;)_z & i(\hat{s}_\| + \hat{s}_\perp\gamma)_y & -i(\hat{s}_\| + \hat{s}_\perp\gamma)_x & 0 \end{bmatrix} \tag{17.86}$$

$$\sigma^{\mu\nu} \;=\; \frac{i}{2}\,(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$$

| | | | |
|---|---|---|---|
| $\hat{s}_\|$ | $=$ | $(\;\hat{\beta}\;\cdot\;\hat{s}\;)\;\hat{\beta}$ | parallel component with regard to $\vec{\beta}$ |
| $\hat{s}_\perp$ | $=$ | $(\;\hat{\beta}\times\hat{s}\;)\times\hat{\beta}$ | orthogonal component with regard to $\vec{\beta}$ |
| $\hat{s}_\otimes$ | $=$ | $(\;\hat{\beta}\times\hat{s}\;)$ | $90^o$ rotated orthogonal component |