Chapter 19
Operators and Observables of the Dirac field
–
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Understanding Relativistic Quantum Field Theory

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Contents

Chapter 19

Operators and Observables of the Dirac field

19.1 The components of the Dirac field

Having studied the propagation mechanism of the Dirac field in the previous chapters using the chiral representation we can separate the most general form of the Dirac field in the following main components.

$$\begin{pmatrix} \xi_L \\ \xi_R \end{pmatrix} e^{i\phi(t,\vec{x})} \;\; Q(t,\vec{x}) \tag{19.1}$$

Where $Q(t,\vec{x})$ is the scalar real valued amplitude field and $|Q|^2$ determines the density, the exponent $\exp(\; i\phi(t,\vec{x})\;)$ defines the local phase of the wave, and the bi-spinor ($\xi_L,\; \xi_R$) describes the chiral components. The bi-spinor is normalized locally as.

$$\overline{\begin{pmatrix} \xi_L \\ \xi_R \end{pmatrix}}\begin{pmatrix} \xi_L \\ \xi_R \end{pmatrix} = \begin{pmatrix} \xi_R^* \\ \xi_L^* \end{pmatrix}\begin{pmatrix} \xi_L \\ \xi_R \end{pmatrix} = \xi_R^*\xi_L + \xi_L^*\xi_R \;=\; 2mc^2 \tag{19.2}$$

Note that the overline swaps the left and richt chiral components besides taking the complex conjugate values. From the same local bi-spinor we can obtain the local energy/momentum or charge/current density

$$\overline{\begin{pmatrix} \xi_L \\ \xi_R \end{pmatrix}}\gamma^\mu\begin{pmatrix} \xi_L \\ \xi_R \end{pmatrix} = \xi_R^*\sigma^\mu\xi_R + \xi_L^*\tilde{\sigma}^\mu\xi_L \;=\; 2(\;E,\; p_xc,\; p_yc,\; p_zc\;) \tag{19.3}$$

$$\text{with} \qquad \sigma^\mu = (\;\sigma^t,\sigma^x,\sigma^y,\sigma^z\;), \quad \text{and} \quad \tilde{\sigma}^\mu = (\;\sigma^t,-\sigma^x,-\sigma^y,-\sigma^z\;)$$

The above values are known locally at each point so that we can derive a local physical velocity (v_x,v_y,v_z) for each point since $v_i = c^2p_i/E$. Internal consistency requires that the local phase can be expanded for an infinitesimal environment like.

$$e^{i\phi(t,\vec{x})} \;=\; e^{-iEt/\hbar \;+\; i\vec{p}\cdot\vec{x}/\hbar \;+\; i\alpha} \tag{19.4}$$

Which provides a second way to determine the four-momentum locally at each point by means of the phase change rates. (which can be determined independently from the magnitude change rates). Both the spinor components as well as the phase change rates must transform consistently going from one reference frame to another. The four-momentum derived from both of them must be equal.

19.2 The total mass/charge operator

This is the simplest operator possible. The mass and charge are Lorentz scalars. To obtain a Lorentz scalar via an integration over space we need compensate for the Lorentz contraction. The bilinear covariant which does this is the mass/charge-density $\bar{\psi}\gamma^0\psi$.

$$m = \frac{1}{2c^2}\int dx^3\ \bar{\psi}\gamma^0\psi \tag{19.5}$$

$$q_{tot} = \frac{q}{2mc^2}\int dx^3\ \bar{\psi}\gamma^0\psi \tag{19.6}$$

The operators are merely constants which adjust the normalization. The mass/charge-density $\bar{\psi}\gamma^0\psi$ is generally normalized as $2mc^2$, using the rest-mass energy.

19.3 The general form of the operators

More involved operators act on the wave-function ψ. They can act on the bi-spinor ξ using γ-matrices or on the phase/amplitude component using differential operators. If $\tilde{\mathcal{O}}$ is the operator then the general form is.

$$\mathcal{O} = \frac{C_{norm}}{2mc^2}\int dx^3\ \bar{\psi}\gamma^0\tilde{\mathcal{O}}\psi \tag{19.7}$$

From the notation used above we can recognize that the operator acts on a Dirac field. We can also use another equivalent notation which resembles the way operators are applied in the non-relativistic Scrödinger theory.

$$\mathcal{O} = \frac{C_{norm}}{2mc^2}\int dx^3\ \psi^*\tilde{\mathcal{O}}\psi \tag{19.8}$$

The equivalence becomes clear when we write this out in more detail. The γ^0 undoes the swap between the left and right chiral components of $\bar{\psi}$

$$\overline{\begin{pmatrix} \xi_L \\ \xi_R \end{pmatrix}}\gamma^0 \; \tilde{\mathcal{O}} \begin{pmatrix} \xi_L \\ \xi_R \end{pmatrix} = \begin{pmatrix} \xi_L^* \\ \xi_R^* \end{pmatrix} \tilde{\mathcal{O}} \begin{pmatrix} \xi_L \\ \xi_R \end{pmatrix} \tag{19.9}$$

19.4 Space-like integration and Special Relativity

In the examples following in this chapter we will use the general form given below, where the $\tilde{\mathcal{O}}$ is a "local density" operator on the field.

$$\mathcal{O} \;\; = \;\; \frac{C_{norm}}{2mc^2} \int dx^3 \; \psi^* \tilde{\mathcal{O}} \psi \tag{19.10}$$

Local because $\psi^* \tilde{\mathcal{O}} \psi$ yields a result for each point and *density* because it acts on the "Lorentz compressed" field: $\psi^*\psi$ is larger (denser) by a factor γ as $\bar{\psi}\psi$ which compensates for the Lorentz contraction of the total field during integration.

There is a fundamental problem however with the spatial integration with which the reader will be familiar with from the version of expression in non-relativistic Quantum Mechanics.

The issue is that physical space-like integration is *forbidden* in Special Relativity: It propagates information instantaneously from $x = -\infty$ to $x = +\infty$. We need to be aware of the limitations of the above expression.

If the field represents a static time-independent wave-function then there is no physical objection to the integration. We can in general expect the result of the integration to be the same in different reference frames (hyperplanes).

However, if there is interaction and the field changes then we can not expect the result of the integration to be the same in different reference frames.

19.5 The total mass/charge current operator

The second simplest operator possible gives us the total electric current or the total mass-flux. These represent the three spatial components of four-vectors which have the Lorentz invariant mass and charge as zeroth component.

We simply use the current-density $\bar{\psi}\gamma^i\psi$ which forms a four-vector together with $\bar{\psi}\gamma^0\psi$.

$$\Phi_{mass} = \frac{1}{2c^2}\int dx^3\ \bar{\psi}\gamma^i\psi \tag{19.11}$$

$$J^i = \frac{q}{2mc^2}\int dx^3\ \bar{\psi}\gamma^i\psi \tag{19.12}$$

We find the operators $\tilde{\mathcal{O}}$ if we rewrite these in the general form $\bar{\psi}\gamma^0\tilde{\mathcal{O}}\psi$ discussed above.

$$\bar{\psi}\gamma^i\psi \;=\; \bar{\psi}\gamma^0\gamma^0\gamma^i\psi \;=\; \psi^*\tilde{\mathcal{O}}\psi \tag{19.13}$$

$$\Downarrow$$

$$\tilde{\mathcal{O}}^i \;=\; \gamma^0\gamma^i \;=\; \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \tag{19.14}$$

19.6 The Hamiltonian time derivative operator

We define the Hamiltonian operator as the first order time-derivative in order to be able to use it as a commutator to obtain the time derivatives of other observables. We recall the Chiral 4d Dirac equation.

$$\left[\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\frac{\partial}{\partial t} + c\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\frac{\partial}{\partial r^i}\right]\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \frac{mc^2}{i\hbar}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{19.15}$$

So we can define the Hamiltonian operator by multiplying all terms with the factor $ic\hbar\gamma^0$.

$$\tilde{\mathrm{H}} \;=\; i\hbar\frac{\partial}{\partial t} \;=\; -\,ic\hbar\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}\frac{\partial}{\partial x^i} + \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} mc^2 \qquad (19.16)$$

We can write the total Hamiltonian in an integral form as follows.

$$H \;=\; \int dx^3 \;\; \mathcal{H} \;=\; \frac{1}{2mc^2}\int dx^3 \;\; \psi^*\left(i\hbar\frac{\partial}{\partial t}\right)\psi \qquad (19.17)$$

Where $\mathcal{H}$ is the Hamiltonian density. Note that the time-derivative only acts on the exponential component of ψ. It is sandwiched between $\bar{\psi}\gamma^0\psi$ where it acts as the Hamiltonian-density operator.

It acts as a *density* operator because the bilinear covariant $\bar{\psi}\gamma^0\psi$ transforms as the charge *density*. Note that the Lorentz transform of the charge density is expressed by the spinor product $\xi\gamma^0\xi$, where $\psi{=}\xi\exp(i\psi)$. It is the relativistic spinor transform rather than the transform of the phase change rates which determines the transform of the density. The density has to increase by a factor of γ due to the Lorentz contraction.

If ψ is a solution of the Dirac equation then we can express the Hamiltonian density as both sides of (19.16)

$$\mathcal{H} \;=\; \frac{1}{2mc^2}\psi^*\left(i\hbar\frac{\partial}{\partial t}\right)\psi \;=\; \frac{1}{2mc^2}\psi^*\gamma^0\left(-ic\hbar\gamma^i\frac{\partial}{\partial x^i} \;+\; mc^2\;\right)\psi \qquad (19.18)$$

We should be able now to obtain the time derivative of an operator $\tilde{\mathcal{O}}$ by means of a commutation with the Hamiltonian operator $\tilde{H}$ as follows.

$$\frac{\partial\tilde{\mathcal{O}}}{\partial t} \;=\; \frac{i}{\hbar}\left[\;\tilde{H}\;,\;\tilde{\mathcal{O}}\;\right] \;=\; \frac{i}{\hbar}\left(\tilde{H}\tilde{\mathcal{O}} \;-\; \tilde{\mathcal{O}}\tilde{H}\right) \qquad (19.19)$$

Where the Hamiltonian operator $\tilde{H}$ is given by.

$$\tilde{H}\;\;(=i\hbar\partial_t) \;\;=\;\; -\,ic\hbar\gamma^0\gamma^i\partial_i \;+\; \gamma^0 mc^2 \qquad (19.20)$$

19.7 The average position operator

The average position operator can be defined as well by using the mass/charge density field $\bar{\psi}\gamma^0\psi$.

$$\vec{x}_{avg} \;=\; \frac{1}{2mc^2}\int dx^3\ \vec{x}\ \ \bar{\psi}\gamma^0\psi \qquad (19.21)$$

The mass/charge density increases as γ with the velocity because the area which confines the wave-function decreases with γ due to the Lorentz contraction. Both effects cancel when the density is integrated over space. The total mass/charge is Lorentz invariant.

$$\vec{x}_{avg} \;=\; \frac{1}{2mc^2}\int dx^3\ \ \psi^*\vec{x}\ \psi \qquad (19.22)$$

Here we have sandwiched $\vec{x}$ as an operator between $\bar{\psi}\gamma^0\psi$ as a "position-density" operator, which simply means that it yields the average position when integrated over space. We can place $\vec{x}$ anywhere since the three components of $\vec{x}$ act as constants in the (three) expressions for the components of $\bar{\psi}\gamma^0\vec{x}\ \psi$.

19.8 The average velocity operator

Having obtained a position operator, we are now able to derive a velocity operator by commutating it with the Hamiltonian operator. Recalling the position and Hamiltonian operators.

$$\tilde{\mathrm{X}}^i \;=\; x^i \qquad (19.23)$$

$$\tilde{\mathrm{H}} \;=\; -ic\hbar\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}\frac{\partial}{\partial x^i} + \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} mc^2 \qquad (19.24)$$

The commutator of these two becomes a constant matrix for each of the three components.

$$\tilde{\mathrm{V}}^i\psi \;=\; \frac{i}{\hbar}\left[\ \tilde{H},\tilde{X}^i\ \right]\psi \;=\; c\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}\psi \qquad (19.25)$$

The mass term did commute but the differential term didn't. Consistent with the other operators we now write for the average velocity.

$$\vec{v}_{avg} \;=\; \frac{c}{2mc^2}\int dx^3 \;\; \psi^* \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \psi \tag{19.26}$$

This is just. (see equation (19.14))

$$\vec{v}_{avg} \;=\; \frac{c}{2mc^2}\int dx^3 \;\; \bar{\psi}\gamma^i\psi \;=\; \frac{c}{2mc^2}\int dx^3 \; j^i \tag{19.27}$$

Where j^i is the current density. The current density increases by an extra factor γ depending on the speed because the volume containing the moving electron becomes smaller by a factor γ due to the Lorentz contraction. The integral over space of the current density gives the total current as we did see with the current operator in section (19.5). The total current J^i is related to the total charge Q by the relation

$$\vec{J} \;=\; \vec{v}_{avg}\; Q \tag{19.28}$$

demonstrating that (19.25) represents the correct velocity operator.

19.9 The acceleration operator

Having shown that the velocity operator yields the correct results when applied properly we can derive an acceleration operator with the use of the Hamiltonian.

$$\tilde{A}^i \;=\; \frac{\partial \tilde{V}^i}{\partial t} \;=\; \frac{i}{\hbar}\left[\; \tilde{H},\; \tilde{V}^i \;\right] \tag{19.29}$$

The velocity operator anti commutes which each term of the Hamiltonian with the exception of the momentum in the same direction. The commutation with the mass term of the Hamiltonian gives.

$$\frac{i}{\hbar}\left[\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} mc^2 \;,\; c\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}\right] \;=\; \frac{imc^3}{\hbar}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \qquad (19.30)$$

While the commutation with the momentum derivative terms of the Hamiltonian results in.

$$\begin{aligned} &\frac{i}{\hbar}\left[-ic\hbar\begin{pmatrix} -\sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix}\frac{\partial}{\partial x^j} \;,\; c\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}\right] \\ &\qquad = \; \varepsilon^{ijk}c\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}\frac{\partial}{\partial x^j} \end{aligned} \qquad (19.31)$$

In more verbose notation we can write for $\tilde{A}^i$.

$$\tilde{A}^i \;=\; \frac{i}{\hbar}\left[\,\tilde{H},\;\tilde{V}^i\,\right] \;=\; c\;\Sigma^k\times\partial_j \;+\; \frac{imc^3}{\hbar}\gamma^i \qquad (19.32)$$

Where Σ^k is the 4x4 matrix with two σ^k on its main diagonal. Sandwiching this between the field ψ gives us.

$$\psi^*\tilde{A}^i\psi \;=\; c\;\bar{\psi}\left(\gamma^i\gamma^5\times\partial_i\right)\psi \;+\; \frac{imc^3}{\hbar}\;\bar{\psi}\gamma^0\gamma^i\psi \qquad (19.33)$$

We should expect that $\psi^*\tilde{A}^i\psi$ is zero for a plane wave so we will study the above expression with this in mind. We find that the above expression transforms like the Lorentz Force.

$$\psi^*\tilde{A}^i\psi \;=\; \frac{i}{m}\frac{d\vec{p}}{dt} \;=\; \frac{iq}{m}\left(\vec{v}\times\vec{B} \;+\; \vec{E}\right) \qquad (19.34)$$

With the $\vec{E}$ component zero in the rest frame and $\vec{B}$ pointing in the same direction as the spin in the rest-frame. We therefor associate $\vec{E}$ and $\vec{B}$ with the magnetization and polarization of the field itself due to the magnetic moment.

$$\psi^*\tilde{A}^i\psi \;=\; \frac{i}{m}\frac{d\vec{p}}{dt} \;=\; \frac{iq}{m}\left(\vec{v}\times\mu_o\vec{M} \;+\; \frac{1}{\epsilon_o}\vec{P}\right) \qquad (19.35)$$

The polarization of the field is the result of the Lorentz transform of the magnetic moment if the velocity is not equal to zero. This Lorentz force is zero in the case of the plane-wave. It the particle is at rest then only the magnetization is non-zero, there is no Lorentz force.

The Lorentz Force evidently stays zero after transformation. The forces from the magnetization and the polarization cancel. In other words:
A plane wave doesn't experiences acceleration.

We now return our attention to the result obtained sofar.

$$\psi^* \tilde{A}^i \psi \;\; = \;\; c\, \bar{\psi} \left(\gamma^i \gamma^5 \times \partial_i\right) \psi \; + \; \frac{imc^3}{\hbar} \; \bar{\psi}\gamma^0\gamma^i\psi \tag{19.36}$$

First we recognize $\bar{\psi}\gamma^k\gamma^5\psi$ as the three spatial components of the axial vector current of the Dirac electron.

$$j_A^i \;=\; \bar{\psi}\gamma^k\gamma^5\psi \tag{19.37}$$

Secondly we see that $\bar{\psi}\partial_j\psi$ corresponds with the momentum of the plane wave. It acts only on the phase change rates of the exponential since the density and the bi-spinor are considered constants throughout the plane-wave.

$$p^i \;=\; -\,\frac{i\hbar}{2mc^2}\;\bar{\psi}\partial_j\psi \tag{19.38}$$

Thirdly we recognize $\bar{\psi}\gamma^0\gamma^i\psi$ from the Gordon decomposition of the vector current as the polarization P^i of the electron field. The polarization is the result of the magnetic moment of a boosted electron. The magnetic moment is Lorentz transformed into an electric dipole moment.

$$P^i \;=\; -\,\frac{q}{\epsilon_o}\,\frac{1}{2mc^2}\;\bar{\psi}\gamma^0\gamma^i\psi \;=\; -\,\frac{q}{\epsilon_o}\,\frac{1}{2mc^2}\;\bar{\psi}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}\psi \tag{19.39}$$

The Gordon decomposition involved the matrix $\sigma^{\mu\nu}$ which contains the matrix-products $\gamma^0\gamma^i$ used above in the zeroth row and column.

$$\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^i, \gamma^j\right] =$$

$$\begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} \sigma^x & 0 \\ 0 & -\sigma^x \end{bmatrix} & \begin{bmatrix} \sigma^y & 0 \\ 0 & -\sigma^y \end{bmatrix} & \begin{bmatrix} \sigma^z & 0 \\ 0 & -\sigma^z \end{bmatrix} \\ \begin{bmatrix} -\sigma^x & 0 \\ 0 & \sigma^x \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} i\sigma^z & 0 \\ 0 & i\sigma^z \end{bmatrix} & \begin{bmatrix} -i\sigma^y & 0 \\ 0 & -i\sigma^y \end{bmatrix} \\ \begin{bmatrix} -\sigma^y & 0 \\ 0 & \sigma^y \end{bmatrix} & \begin{bmatrix} -i\sigma^z & 0 \\ 0 & -i\sigma^z \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} i\sigma^x & 0 \\ 0 & i\sigma^x \end{bmatrix} \\ \begin{bmatrix} -\sigma^z & 0 \\ 0 & \sigma^z \end{bmatrix} & \begin{bmatrix} i\sigma^y & 0 \\ 0 & i\sigma^y \end{bmatrix} & \begin{bmatrix} -i\sigma^x & 0 \\ 0 & -i\sigma^x \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \tag{19.40}$$

It was shown that the bilinear $\bar{\psi}\sigma^{\mu\nu}\psi$ can be associated with the magnetization/ polarization tensor of the electron's field due to the magnetic moment.

$$\mathbb{M}^{\mu\nu} = -\frac{q}{\epsilon_0}\frac{1}{2mc^2}\,\bar{\psi}\sigma^{\mu\nu}\psi = \begin{bmatrix} 0 & -\ P_x & -\ P_y & -\ P_z \\ P_x & 0 & -\mathrm{ic}M_z & \mathrm{ic}M_y \\ P_y & \mathrm{ic}M_z & 0 & -\mathrm{ic}M_x \\ P_z & -\mathrm{ic}M_y & \mathrm{ic}M_x & 0 \end{bmatrix} \tag{19.41}$$

So, we have associated the second term at the right hand side of equation (19.36) with the polarization of the field due to the transformation of its own magnetic moment.

$$\frac{imc^3}{\hbar}\,\bar{\psi}\gamma^0\gamma^i\psi \quad \text{is identical to} \quad \frac{iq}{\epsilon_o m}\,\vec{P} \tag{19.42}$$

We now need to associate the first term at the right hand side of equation (19.36) with $\vec{v}\times\vec{M}$.

$$c\,\bar{\psi}\left(\gamma^i\gamma^5\times\partial_i\right)\psi \quad \text{is identical to} \quad \frac{iq\mu_o}{m}\left(\vec{v}\times\vec{M}\right) \tag{19.43}$$

In a plane-wave everything is constant except the phase $\exp(-iEt/\hbar + ipx)$. The differential operator act only on the phase so we can separate the two operators below.

$$j_A^i \;=\; \bar{\psi}\gamma^k\gamma^5\psi \qquad\qquad p^i \;=\; -\frac{i\hbar}{2mc^2}\;\bar{\psi}\partial_j\psi \tag{19.44}$$

The term is thus proportional to the cross product of p^i and j_A^i. What we need to show is that.

$$v^i \times M^i \quad\propto\quad p^i \times j_A^i \tag{19.45}$$

And this has to hold in all reference frames. The magnetization M^i and the axial current density j_A^i point in the same direction as the spin in the rest-frame, however. The transform differently. The first transforms like as a combination of tensor component while j_A^i transforms as an axial vector.

The transform of both M^i and j_A^i depend on the relation of the pointer to the boost. The orthogonal and parallel components transform differently. Here we only need to take the orthogonal components into account due to the cross-product with the velocity/momentum, and indeed, the orthogonal component of M^i transforms with a factor γ more as the orthogonal component of j_A^i so, equation (19.45) holds.

$$\psi^*\tilde{A}^i\psi \;=\; \frac{1}{m}\frac{d\vec{p}}{dt} \;=\; \frac{q}{m}\left(\vec{v}\times\mu_o\vec{M} \;+\; \frac{1}{\epsilon_o}\vec{P}\right) \tag{19.46}$$

In order to make the relation to hold we have used an extra condition here, namely that both the bi-spinor and the phases are identically Lorentz transformed. That they have both the same velocity. Without this assumption the plane wave would have a non-zero acceleration.

19.10 The intrinsic spin operator

The operator $\tilde{S}^i$ for the intrinsic spin of the Dirac particle is given by.

$$\tilde{S}^i \;=\; \hbar\, \gamma^0\gamma^i\gamma^5 \;=\; \hbar\, \Sigma^i \tag{19.47}$$

Where we can write out terms $\gamma^0\gamma^i\gamma^5$ and Σ^i explicitly as.

$$\tilde{S}^i = \hbar \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} = \hbar \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \tag{19.48}$$

The matrix Σ^i owns its name to the σ^i matrices on the diagonal. The operator acts on the bi-spinor part of ψ. Applying the operator $\tilde{S}^i$ gives us the total spin integrated over space.

$$\begin{aligned} s^i \;&=\; \frac{\hbar}{2mc^2}\int dx^3 \;\; \psi^*\Sigma^i\psi \;=\; \frac{\hbar}{2mc^2}\int dx^3 \;\; \bar{\psi}\gamma^i\gamma^5\;\psi \\ &\Downarrow \\ s^i \;&=\; \frac{\hbar}{2mc^2}\int dx^3\; j^i_A \end{aligned} \tag{19.49}$$

The quantity j^i_A=$\bar{\psi}\gamma^i\gamma^5\;\psi$ is the axial current density as we have seen in the chapter on the bilinear covariant fields of the Dirac particle. The spin s^i and the axial current density j^i_A always point to the same direction, independent of the reference frame.

The size of the components of j^i_A increase with an extra factor of γ because it is a density and the volume of the field decreases by a factor γ. If $\hat{s}$ represents the unit spin-vector at rest, and with $\hat{s}_\perp$ and $\hat{s}_\parallel$ being the components of $\hat{s}$ orthogonal and parallel to the boost $\vec{\beta}$, then we can write the transform as.

$$\bar{\psi}\gamma^\mu\gamma^5\psi \;=\bar{\psi}\psi \begin{bmatrix} \gamma(\hat{s}\cdot\vec{\beta}) \\ (\hat{s}_\perp + \hat{s}_\parallel\gamma)_x \\ (\hat{s}_\perp + \hat{s}_\parallel\gamma)_y \\ (\hat{s}_\perp + \hat{s}_\parallel\gamma)_z \end{bmatrix} \tag{19.50}$$

19.11 The orbital angular momentum operator

The operator for the orbital angular momentum is analogous to the classical version $\vec{L} = \vec{r} \times \vec{p}$ and obtained by the substitution of p^i with $-i\hbar\partial_i$.

$$\tilde{L}^i = -i\hbar\vec{r} \times \frac{\partial}{\partial x^i} = -i\hbar \begin{pmatrix} y\partial_z - z\partial_y \\ z\partial_x - x\partial_z \\ x\partial_y - y\partial_x \end{pmatrix} \tag{19.51}$$

This operator acts on the phase/amplitude components of ψ via the differential operator. The total orbital angular momentum integrated over space is given by.

$$L^i = \frac{1}{2mc^2} \int dx^3 \;\; \psi^* \left(-i\hbar\vec{r} \times \partial_i\right) \psi \tag{19.52}$$

The outer product with $\vec{r}$ can be brought outside the ψ "sandwich".

$$L^i = \frac{-i\hbar}{2mc^2} \int dx^3 \;\; \vec{r} \times \left(\psi^* \partial_i \psi\right) \tag{19.53}$$

19.12 Conservation of total angular momentum

When Dirac first published his theory in the beginning of 1928, [1],[2] he already had made an important discovery concerning the spin 1/2 nature of the electron.

Studying the conservation of the orbital and spin angular momentum he found that the two were not independently conserved. He commutated the two operators with the Hamiltonian to obtain the time derivatives and found.

$$\frac{\partial \tilde{L}^i}{\partial t} = \frac{i}{\hbar}\left[\, \tilde{H},\ \tilde{L}^i \,\right] = ic\hbar\,\gamma^0 \begin{pmatrix} \gamma^y\partial_z - \gamma^z\partial_y \\ \gamma^z\partial_x - \gamma^x\partial_z \\ \gamma^x\partial_y - \gamma^y\partial_x \end{pmatrix} \tag{19.54}$$

$$\frac{\partial \tilde{S}^i}{\partial t} = \frac{i}{\hbar}\left[\, \tilde{H},\ \tilde{S}^i \,\right] = -2ic\hbar\,\gamma^0 \begin{pmatrix} \gamma^y\partial_z - \gamma^z\partial_y \\ \gamma^z\partial_x - \gamma^x\partial_z \\ \gamma^x\partial_y - \gamma^y\partial_x \end{pmatrix} \tag{19.55}$$

Which we can write more condensed by using the cross product of the γ matrices and the partial derivatives.

$$\frac{\partial \tilde{L}^i}{\partial t} = ic\hbar\,\gamma^0\left(\gamma^i \times \partial_i\right), \qquad \frac{\partial \tilde{S}^i}{\partial t} = \; -2ic\hbar\,\gamma^0\left(\gamma^i \times \partial_i\right) \tag{19.56}$$

Which shows that neither of the two is locally conserved. The similarity between the two however leads directly to an expression for the total angular momentum operator which is locally conserved.

$$\tilde{J}^i \;=\; \tilde{L}^i \;+\; \tfrac{1}{2}\tilde{S}^i \tag{19.57}$$

The time derivative of this operator obtained via the Hamilton is locally zero, so J^i is a conserved quantity, or in other words "*a constant of the motion*".

$$\frac{\partial \tilde{J}^i}{\partial t} \;=\; \frac{i}{\hbar}\left[\;\tilde{H},\;\tilde{J}^i\;\right] \;=\; 0 \tag{19.58}$$

This showed that the electron has an intrinsic magnetic moment of $\frac{1}{2}\hbar$

19.13 Plane wave angular momentum conservation

It turns out, on closer inspection, that in case of a plane-wave both spin and orbital angular momentum *are* in fact locally conserved *independently.*

$$\frac{\partial \tilde{L}^i}{\partial t} = ic\hbar\,\gamma^0\left(\gamma^i \times \partial_i\right), \qquad \frac{\partial \tilde{S}^i}{\partial t} = \; -2ic\hbar\,\gamma^0\left(\gamma^i \times \partial_i\right) \tag{19.59}$$

This is not directly obvious in the expressions above. We first have to let the operators act on the wave-function ψ.

$$ic\hbar\,\psi^*\left(\gamma^0\gamma^i \times \partial_i\right)\psi \;=\; ic\hbar\,\bar{\psi}\left(\gamma^i \times \partial_i\right)\psi \tag{19.60}$$

The derivatives act purely on the phase-change rates of the exponential in the case of a plane-wave while the γ matrices act only on the bi-spinor. We recognize two different currents in the expression above. First there is the Dirac vector field which represents the four-vector momentum of the particle.

$$p^i \;=\; \tfrac{1}{2c}\,\bar{\psi}\gamma^0\gamma^0\gamma^i\psi \;=\; \tfrac{1}{2c}\,\bar{\psi}\gamma^\mu\psi \tag{19.61}$$

Secondly there is the expression which derives the momentum of the plane wave from the phase changes of the complex exponential.

$$p^i \;=\; \tfrac{1}{2mc^2}\,\bar{\psi}\partial_\mu\psi \tag{19.62}$$

The momenta should be equal. Note that first is derived from the bi-spinor and the second from the phase-change-rates using the derivatives. Equation (19.60) represents the cross-product from the two which should be zero.

Note that we used an extra condition here, namely that both the bi-spinor and the phases are identically Lorentz transformed. Although evidently a reasonable assumption, it is only with this extra condition that the spin and orbital angular momenta are conserved separately.

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