Chapter 4
Non-simultaneity from the classical wave equation
–
from my book:
Understanding Relativistic Quantum Field Theory

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Contents

Chapter 4

Non-simultaneity from the classical wave equation

4.1 Changing reference frames and non-simultaneity

There is nothing particular difficult in understanding what happens in a *single* reference frame. Lorentz contraction is predicted for the moving stable solutions of the wave equations for massless and massive objects (See the chapter on the Klein Gordon equation for the latter). The effect of a slower moving clock is perfectly predicted by the fact that signals have to travel a longer path in moving objects.

Nature's magnificent skill's as a perfect illusionist happens when we *change* from one reference frame to another reference frame. Both Lorentz contraction and Time dilation seem to be "undone" and perfectly reversed. After changing to a new reference frame the effects now apply to the previous rest frame.

The trick Nature uses is that of changing simultaneity when changing reference frames. Essential for understanding Special Relativity is understanding simultaneity: The mechanism which lead us to observe a different simultaneity also causes the reversal of Lorentz contraction and Time dilation.

In this chapter we also want to understand this mechanism. Why do we *perceive* different simultanities in different reference frames? Here we will show how this follows in its completeness from the classical physics of propagation. The physics of the wave equations. One can say, on one hand, that these equations *comply* with special relativity. However, on the other hand one can demonstrate that they *give rise* to all of special relativity, and this includes its most puzzling aspect, that of simultaneity.

The single additional postulate we might need to make in classical physics is about the preference of one reference frame above the others. As today, test with accuracies exceeding 1 part in 10^{-17} were not able to single out any reference frame above the others. That is, physics is the same in two different reference with a precision better than that.

Such independency of reference frames assures to us humans that, for instance, live will go on as usual when the Milky Way has rotated over 180 degrees (in approximately 100 million years) and we have to deal with a reference frame which has a speed of 1.6 million kilometers per hour when compared with our current reference frame.

4.2 Lorentz invariance of the wave equation

We did already see the natural occurrence of Lorentz contraction of moving solutions of the classical wave equation. This is easily shown by constraining the classical wave equation to allow only moving, stable solutions. Constrains which do so for a velocity v_x are for instance:

$$\partial_t \Phi \;=\; -\,v_x\,\partial_x \Phi, \qquad \partial_t^2 \Phi \;=\; v_x^2\,\partial_x^2 \Phi \tag{4.1}$$

Doing the differentiation in time ∂_t of the stable, shifting function Φ is the same as doing the differentiation in ∂_x and multiplying by a factor of $-v_x$. The second order constraint can used to remove the time dependency from the wave equation and show the Lorentz contraction in the x-direction.

$$\begin{gathered} \left\{ \tfrac{1}{c^2}\,\partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2 \right\}\Phi \;=\; 0 \\ \Longrightarrow \\ \left\{ \left(1 - \tfrac{v^2}{c^2}\right)\partial_x^2 + \partial_y^2 + \partial_z^2 \right\}\Phi \;=\; 0 \end{gathered} \tag{4.2}$$

The classical wave equation and consequently, its (light cone) propagator are Lorentz invariant. They are independent of the reference frame.

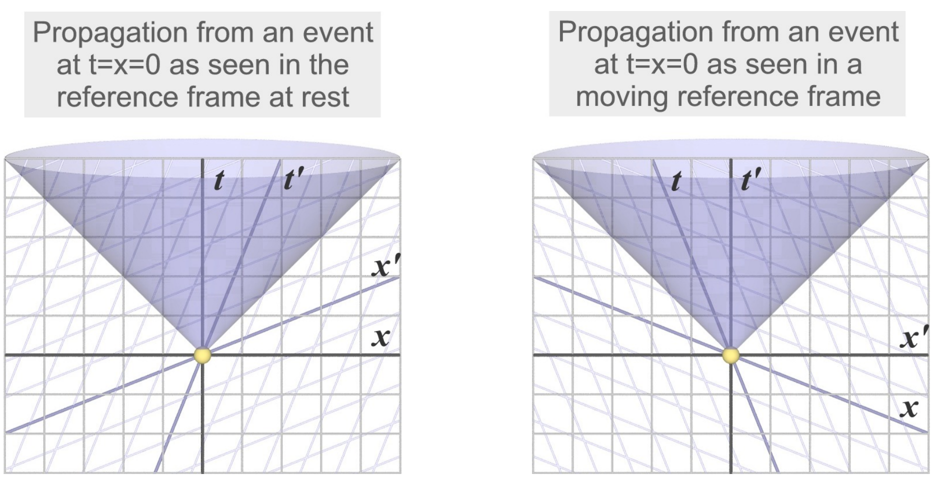

Figure 4.1: Lorentz invariance of the light cone propagator

To verify that the classical wave equation is Lorentz invariant we have to differentiate along the coordinates of the transformed reference frame and to check that at each point the results are the same. (We'll use c=1 here)

$$\left(\partial_{t'}^2 - \partial_{x'}^2 - \partial_{y'}^2 - \partial_{z'}^2\right)\Phi \quad = \quad \left(\partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2\right)\Phi \qquad (4.3)$$

We can do this transformation by using the chain rule to express the transformed differentials in rest frame differentials using the coefficients of the Lorentz transform given by.

$$\left.\begin{array}{r} t' = \gamma(t - \beta x) \\ x' = \gamma(x - \beta t) \end{array}\right\} \quad \Longrightarrow \quad \frac{\partial t}{\partial t'} = \frac{\partial x}{\partial x'} = \gamma, \qquad \frac{\partial x}{\partial t'} = \frac{\partial t}{\partial x'} = -\beta\gamma \qquad (4.4)$$

The 1st and 2nd order derivatives expressed in rest frame differentials are.

$$\begin{array}{rclrcl} \partial_{t'} & = & \gamma\partial_t - \beta\gamma\partial_x, & \partial_{t'}^2 & = & \gamma^2\partial_t^2 - 2\beta\gamma^2\partial_t\partial_x + \beta^2\gamma^2\partial_x^2 \\ \partial_{x'} & = & \gamma\partial_x - \beta\gamma\partial_t, & \partial_{x'}^2 & = & \gamma^2\partial_x^2 - 2\beta\gamma^2\partial_t\partial_x + \beta^2\gamma^2\partial_t^2 \end{array} \qquad (4.5)$$

$$\text{and thus:} \qquad \partial_{t'}^2 - \partial_{x'}^2 \quad = \quad \partial_t^2 - \partial_x^2 \qquad (4.6)$$

The light cone propagator was derived in (**??**) by letting the inverted wave equation operator act on the delta function event at the origin.

$$\begin{array}{rclcl} \left(\partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2\right) \ \mathcal{D}_{(t,x,y,z)} & = & \delta_{(t,x,y,z)} & & \\ & \Longrightarrow & & & \\ \left(\partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2\right)^{-1} \delta_{(t,x,y,z)} & = & \mathcal{D}_{(t,x,y,z)} & = & \frac{\theta(t)}{2\pi}\ \delta(s^2) \end{array} \qquad (4.7)$$

Where $\theta(t)$ is the Heaviside step function and the function $\delta(s^2)$ is non zero on the light cone only. The light cone propagator $\mathcal{D}(x^\mu)$ is Lorentz invariant because the parameter s^2 is Lorentz invariant.

$$s^2 \quad = \quad \left(t^2 - x^2 - y^2 - z^2\right) \quad = \quad \left(t'^2 - x'^2 - y'^2 - z'^2\right) \qquad (4.8)$$

The following sections will use the property of propagation with c on the light cone to show how non-simultaneity follows from the wave equation and classical physics without any need for specific postulates.

4.3 Observed simultaneity and Derived simultaneity

Figure 4.2 shows three equal houses[1]. We will use this image to discuss how we infer simultaneity from events we see happening. The naive observer would infer the simultaneity of events at locations a, b and c if he sees the events happening at the same time.

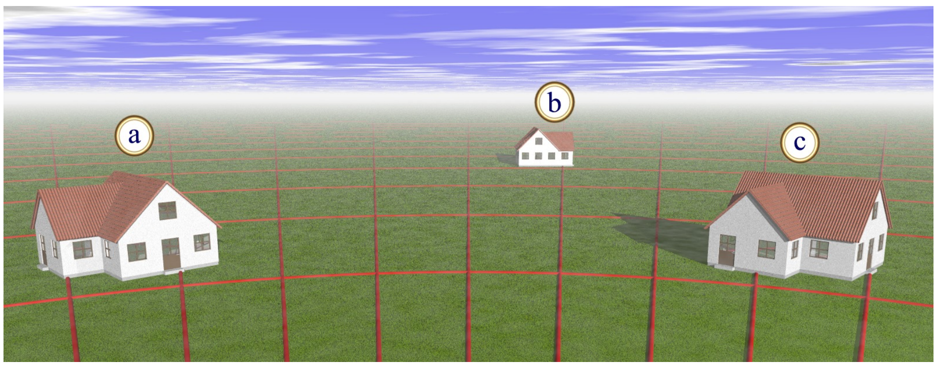

Figure 4.2: simultaneity from equidistant events

With some scientific background he would regard the events at a and c simultaneous because they happen at the same distance while the event at b happened earlier because b is further away.

He considers the locations a and c equidistant because the houses are equally sized. The house at b is considered further away because of it's size. More specifically, the distance is inferred from the angle which the houses occupy his viewing field. We can now define the *Observed* simultaneity as:

The simultaneous arrival of light from events at the same distance, where equidistance is inferred from the viewing angle which objects occupy in the viewing field.

Using the angular size to determine the distance and the assumption of a finite speed of light which is independent of the direction (isotropic velocity), we can define a simultaneity of events at any distance. We call this the *Derived* simultaneity. In this way we can define the entire four dimensional coordinate system of our reference frame.

[1]3D models from Friedrich A. Lohmueller, http://www.f-lohmueller.de

4.4 Everybody sees the same light-cone frame

It's essential to realize that an observer, at a given moment t, acquires information from its environment which is basically independent of its velocity in the sense that, when looking around or taking pictures, the total information gathered at the *focal point* does not depend on the speed.

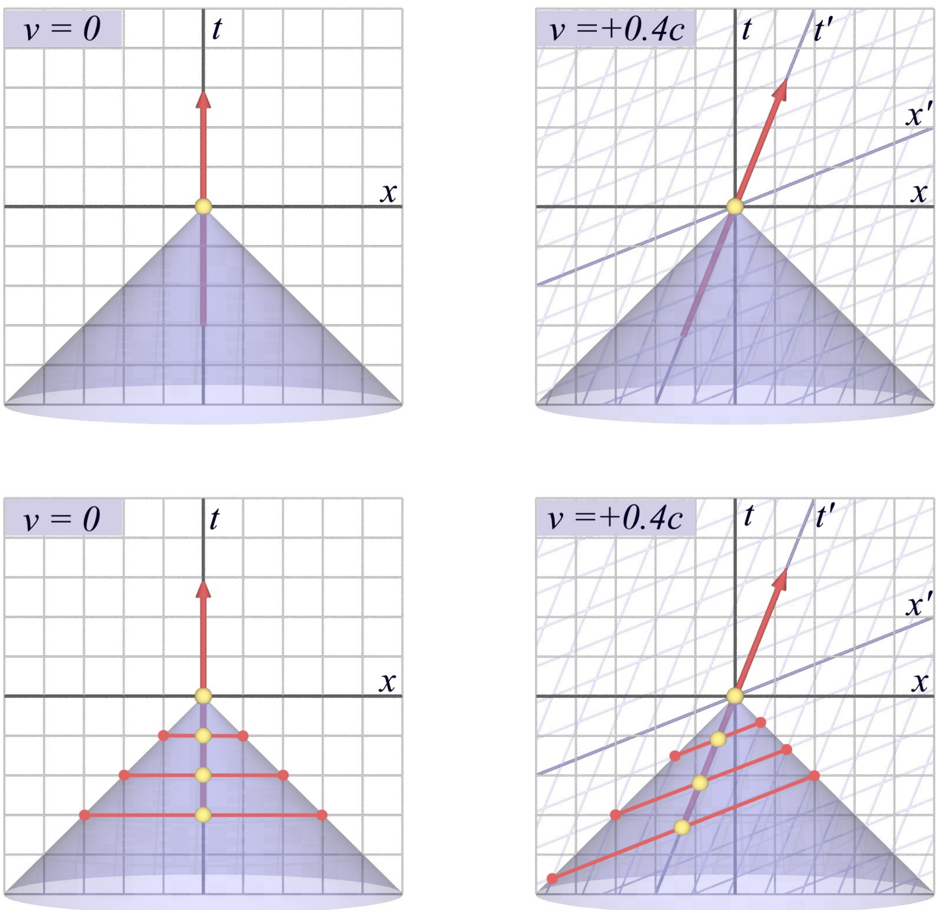

Figure 4.3: 2+1d view of the Light cone

The *focal point* is the *top of the light cone*. The upper images of figure 4.3 show us that information from the same set of events arrives at the same time at the focal point, independent of the velocity of the focal point.

The "frame" which we literally *see* is the *light-cone frame*. We never directly see that what we intuitively call the *now*: the set all of events we define as happening at "the same time". We derive our space-time coordinate system indirectly. We infer it from the way we see the light-cone frame.

Apparently the different ways of seeing the same set of events at the same time leads us to constructing different rest frames and different coordinate systems. Therefor, understanding the illusion of different reference frames amounts to understanding the different ways we see the light-cone frame.

Such differences amount to changes of the apparent angle of the rays coming from an object, changes of the apparent wavelength (color) of the light form objects. However, the total information does not change.

We want to see what happens when our surrounding as a whole is accelerated. Say we could take our world and the solar system and accelerate it to a billion kilometer per hour. We should be able to do so without perceiving any differences.

Why is it that our daily surrounding looks the same as ever, while at the same time we infer that our coordinate system has become very different compared to the restframe? Why is the frame of simultaneity which we infer from how we see the light-cone frame so different?

The bottom images of figure 4.3 relate events on the light-cone with previous locations of the observer. The requirement of symmetry between events at both sides leads us to connect events which are on the x'-axis in the case of the moving observer. The trajectory of the moving observer lays in the middle of the selected events on the lightcone.

If we are to understand non-simultaneity as an emergent result of the wave equations, then we need to explain the *mechanism* which causes the moving observer to *infer* that these events are simultaneous events.

In the previous section we did see from figure 4.2 that the notion of simultaneity is inferred from the *simultaneous* arrival of light from *equidistant* events. Equidistance is inferred when equally sized objects have the same angular size in the viewing field and are therefor projected with the same size on a photograph or on our vision. The further assumption of an isotropic speed of light then allows us to define the complete set of all simultaneous events.

4.5 Passengers in rows and atoms in rows

The simplest configuration to study in different reference frames is possibly an array of equally spaced identical objects. In the case of atoms in a lattice structure one can study the required equilibrium of force between direct neighbors. The example of rows of passengers is useful to study why the passengers accelerated to ultra relativistic velocities do not notice any velocity dependent effects whatsoever when looking around inside their cabin.

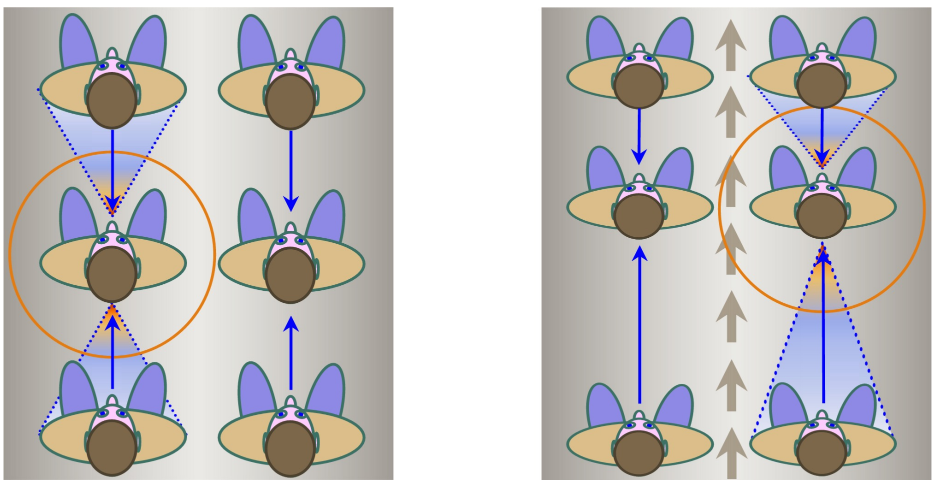

Figure 4.4: passengers in a row

Figure 4.4 shows the cabin at rest on the left side. Light coming from either the front passenger or the passenger at the back takes the same time to reach the passenger in the middle. If the passenger in the middle would take a 360^o degrees panorama picture then it would show an equally sized front seat and back seat passenger. (In our example we use "identical" passengers just for simplicity).

The right hand side of same figure shows a very fast forward moving cabin. Light from the front passenger reaches the middle passenger much faster with a relative speed of $c+v$, while the light from the back passenger goes relatively slower with $c-v$. A 360^o picture taken by the middle passenger at a certain time t would combine a nearby front seat passenger with a further away back seat passenger.

4.6 The velocity dependent viewing angle

At first glance one might expect that the picture would show a larger front seat passenger and a smaller back seat passenger. This because the "light ray cone" from the front seat passenger has a wider spatial angle as the one from the passenger in the back seat.

However, special relativity tells us of course that both passengers should have the *same* size on the picture: Physics is reference frame independent. Now how then does this work? Clearly there must be a counter effect which makes the nearby objects in the front look smaller and the objects at back look bigger.

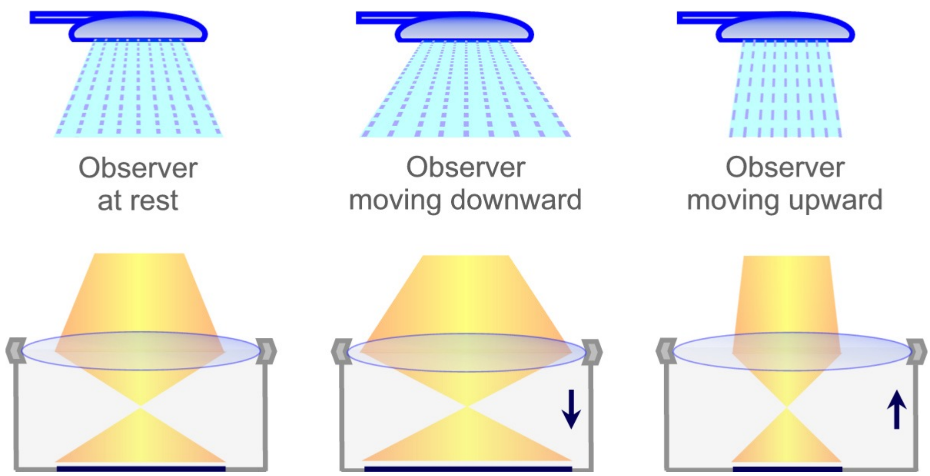

Figure 4.5: Velocity dependent apparent size

The effect is compensated because objects also look larger or smaller, depending on the speed, as the result of another effect: The relative spatial angle under which the rays propagates appears larger or smaller depending on the velocity. The different absolute angles in figure 4.4 will appear the same because the moving observers see different angles.

A situation which can't hardly be more every-day-like is that of the shower in figure 4.5. If we decent bending our knees then the apparent angle of the droplets becomes wider. If we would decent at the same speed as the droplets fall then we would see the droplets moving horizontally away from us in all directions. The angle would be 180^o in this case.

4.7 Simultaneity and the invariance of size

We first want to show that in a more visual way why the apparent size of the passengers before us and behind us is independent of the reference frame. It is the effective *relative* viewing angle which cancels the asymmetries shown in the lower part of figure 4.6.

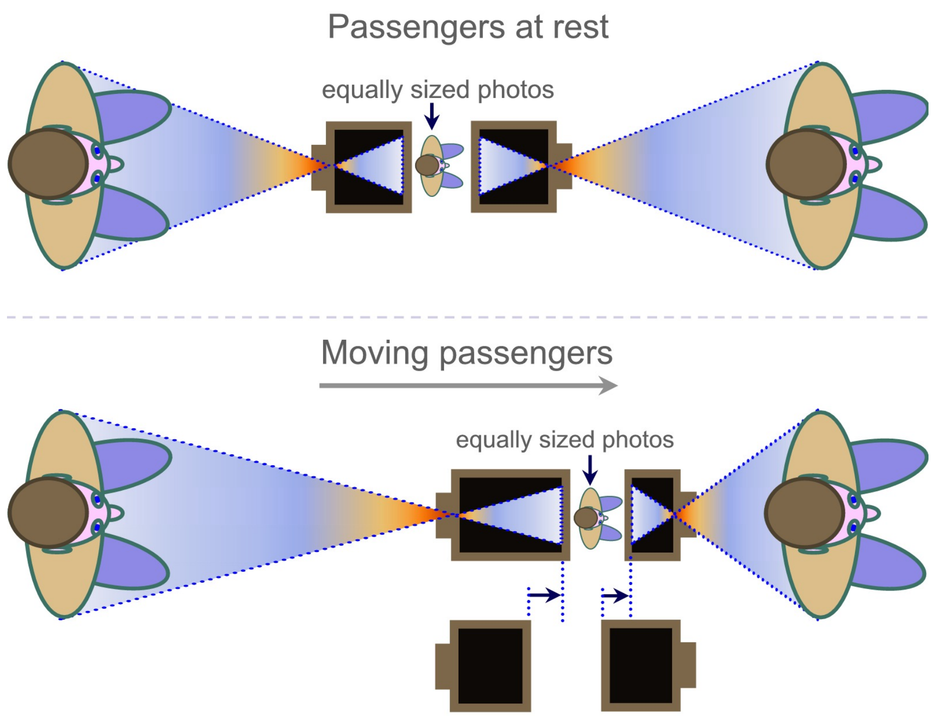

Figure 4.6: Lorentz invariance of the photo size.

The result of the cancelation is that the apparent size becomes Lorentz invariant, independent of the velocity. It does however introduce a shift in the simultaneity as it is inferred by the observer.

Figure 4.6 shows how photos are taken from the passengers. For simplicity we use pinhole "camera obscuras" here. This reduces the complexity of geometry to two ray-cones. One from the object to the point of focus and the second from there to the image sensor.

The light rays from the passenger at the back takes more time to reach the moving camera compared to the light rays from the passenger in front. The photos turn out equal in size (Lorentz invariant) in spite of what the different *absolute* viewing angles of the ray-cones would suggest.

The photos end up equal because the length of the ray-cones inside the cameras is also asymmetric. The image sensor moves away or towards the rays after the focal point is passed.

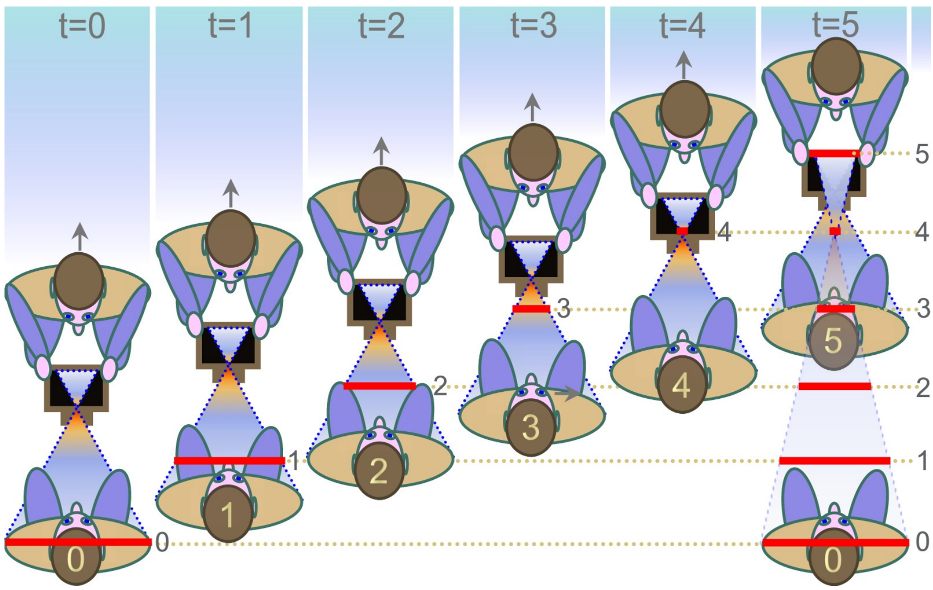

Figure 4.7: Taking a photograph while moving, step by step.

Figure 4.6 shows the starting point of the rays at the passengers, and the end point of the arrays in the camera obscura. These start and end events happen at different times while the passengers and the camera are moving in the mean time .

In figure 4.7 we aim to visualize this step by step. The propagation of the light front from the passenger towards the image sensor is numbered from 0 through 5. The ray cone angles shown are the *relative* angles. At the rightmost site of figure 4.7 we combine all the intermediate wavefronts together and see how they define the *absolute* ray-cone angle in figure 4.6.

4.8 The relative versus the absolute viewing angle

The *absolute* viewing angle φ_{abs} as shown in figure 4.8 is defined by the paths of the light rays to the final position of the observer. The *relative* viewing angle φ_{rel} is based on the relative positions of the observer with regard to the object he is viewing.

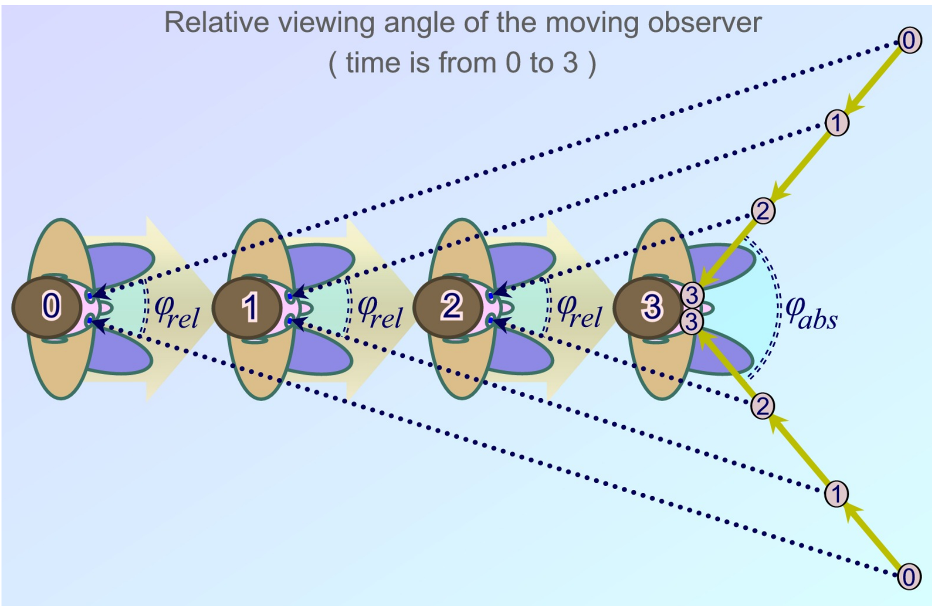

Figure 4.8: The relative angle φ_{rel} versus the absolute angle φ_{abs}

From the intermediate steps we see that the moving passenger sees the light from the object approaching him under the *relative* angle and not under the *absolute* angle. The relative viewing angles of the ray-cones of the passenger in front him and the passenger behind him are the same, while the absolute viewing angles differ.

The observer assumes from the identical sizes (identical relative viewing angles) that both passengers are at equal distances away from him. Furthermore, from the identical sizes and the equal distances he assumes that the rays were emitted at the same time. Hereby he defines a different notion of simultaneity. A simultaneity which is different from that of the observer at rest.

Note that the naive observer bases his notion of simultaneity on the observation that events occur at the same time while the more educated observer will also consider the time delay of the propagation.

However, he assumes that equal objects at equal distances have the same size, and so he will conclude that the light rays took an equal amount of time to reach him, and consequently, that the speed of light is Lorentz invariant and always equal to c.

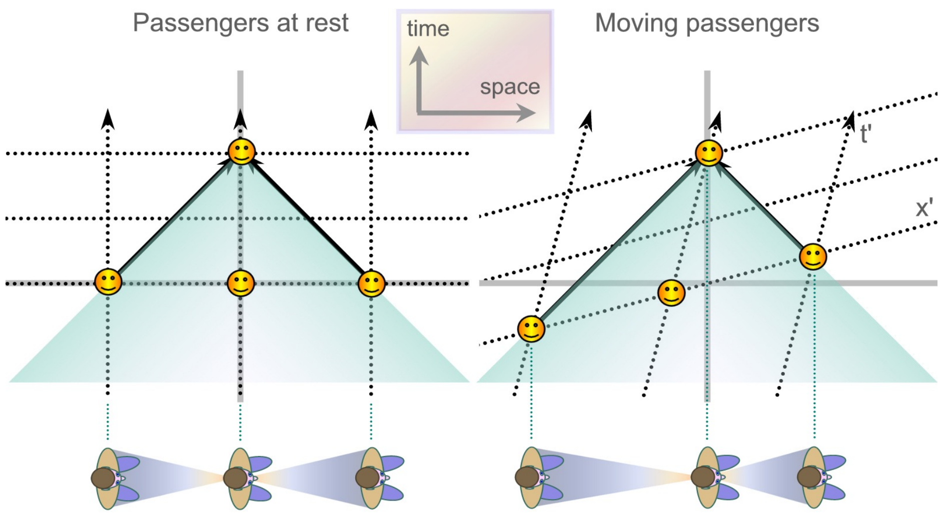

Figure 4.9: Light cone frames of passengers at rest and moving

The difference of simultaneity is the result of the difference in start times of the light rays towards the moving observer . Figure 4.9 show the Minkowski diagrams associated with the situations at rest and moving.

The light-cones are the same in both cases. The arrows show the paths in space-time of the passengers at rest and moving. The crossings of the x-axis and x'-axis mark the events from which the light rays reach the observer in the middle at the at the same time.

From this we can conclude: the x-axis and x'-axis define the planes of simultaneity for the observer at rest and the observer in motion respectively.

Now we add an extra spatial dimension to look at what happens at the left and right sides of the moving observers. Figure 4.10 shows the light-cone from two different viewpoints. It is sliced by various x'-planes which are the planes of simultaneity of the moving observer.

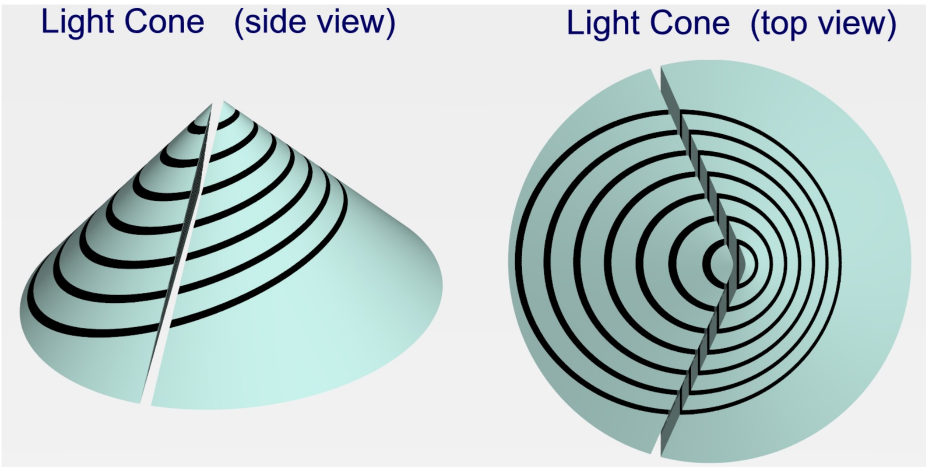

Figure 4.10: Intersections with the 2+1 dimensional light cones

The light-cone is furthermore sliced by a plane spawned by the t'-axis and the line orthogonal to the moving observer. We see from the top view (in the rest frame) that the left and right side of the moving observer (as seen in his own rest-frame) are not under 90^o. This is shown more explicit in figure 4.11 which shows the observer at rest and moving between his left and right side passengers.

Figure 4.12 shows in subsequent time-steps that, even though the light-rays from the side passengers are pointed to the observer from the back, the signal itself is always approaching the moving observer at $\pm 90^o$ angles.

The intersections of planes and cones in figure 4.10 are ellipses. Looking at the cone from the above gives us a top-view projection which shows also ellipses. These are the ellipses of simultaneity, light emitted from any point of the ellipse did reach the center at the same time and appear equidistant.

Light rays at 90^o angles in the passengers restframe.

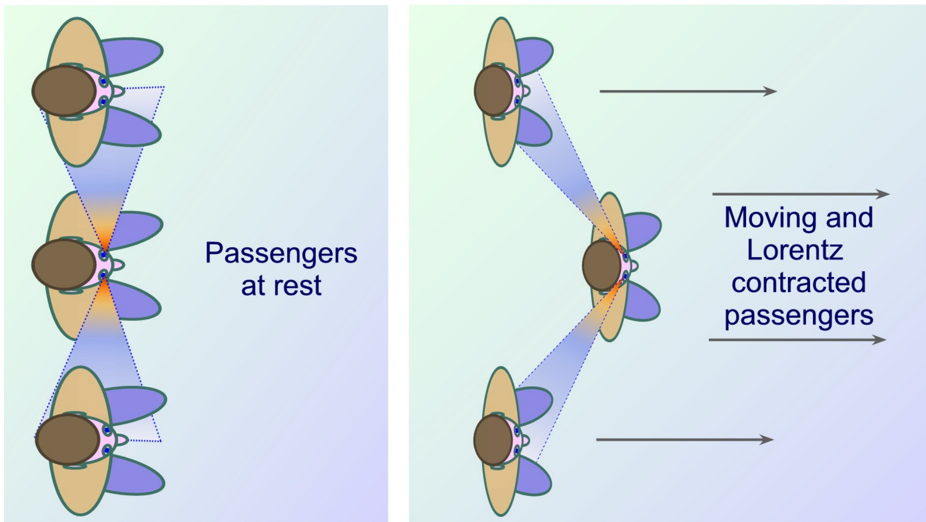

Figure 4.11: Absolute angles of the light rays from side passengers

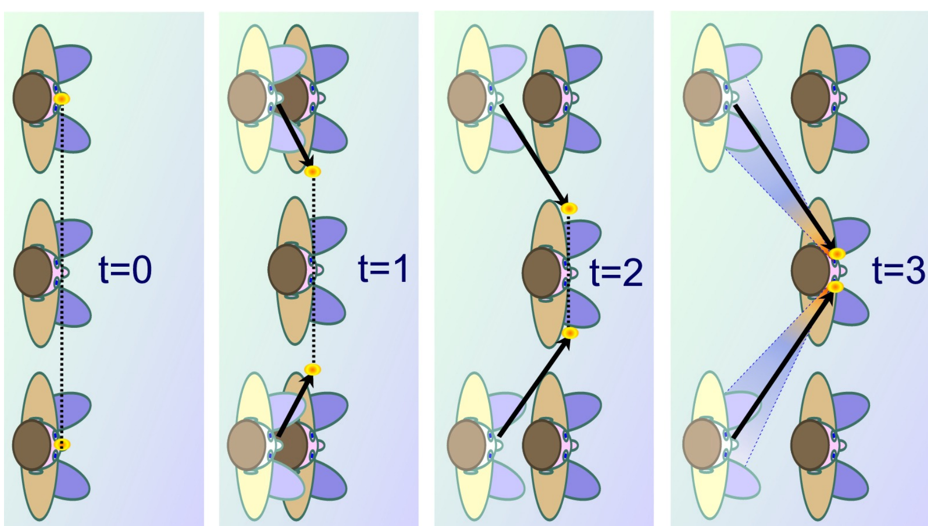

Figure 4.12: light rays from side passengers, step by step

4.9 The ellipsoids of simultaneity

Having spend an unusual long time on visualizations we'll now spend some time to fill in some of the elementary mathematical properties.

The left part of figure 4.13 shows us our moving Lorentz contracted passengers. The observers are Lorentz contracted because the stable solutions of the wave equations which determine the physics of the observers are Lorentz contracted. The wave equations impose Lorentz contraction.

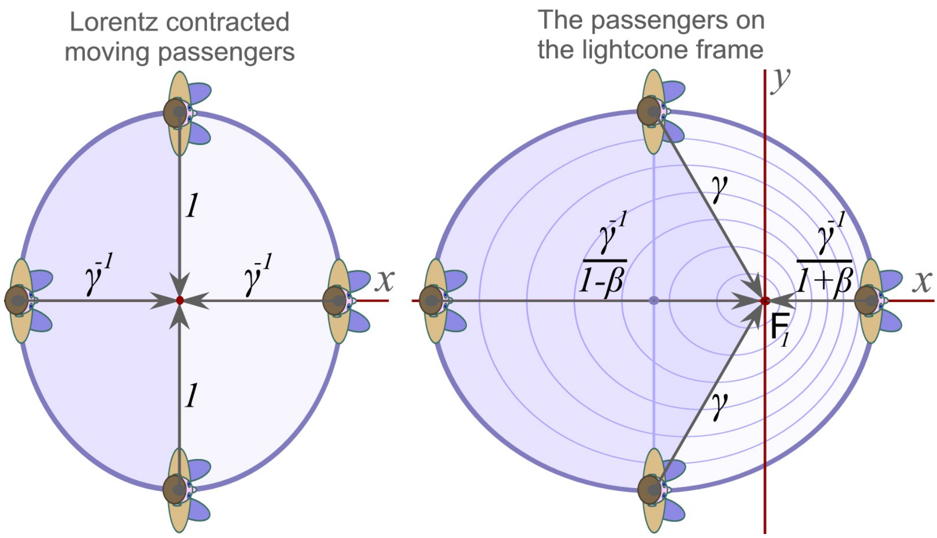

Figure 4.13: light rays from side passengers, step by step

The goal is to understand why these observers see each other as if they were at rest, but, with a different definition of simultaneity. An observer in the center will assume that events are simultaneous as light from equidistant events reach him at the same time, while assuming equidistance if equal objects have equal sizes in his viewing field (equal angles)

To construct an *ellipsoid of simultaneity*, as shown in the right image of figure 4.13, we need to group all events from which light reaches the moving center simultaneously. These events happen at different times on the Lorentz contracted ellipsoid at the left. The moving observer will consider these events as simultaneous.

The two propagation times shown on the x-axis of the ellipsoid of simultaneity in figure 4.13 follow from the relative speeds of the light towards the observer at focal point F_1, which are $1-\beta$ and $1+\beta$. The time delay from the passengers at the sides is determined by the relative velocity $\sqrt{1-\beta^2}$ by which the vertical distance is reduced as light from the side passengers propagates to the moving observer as shown in figure 4.12.

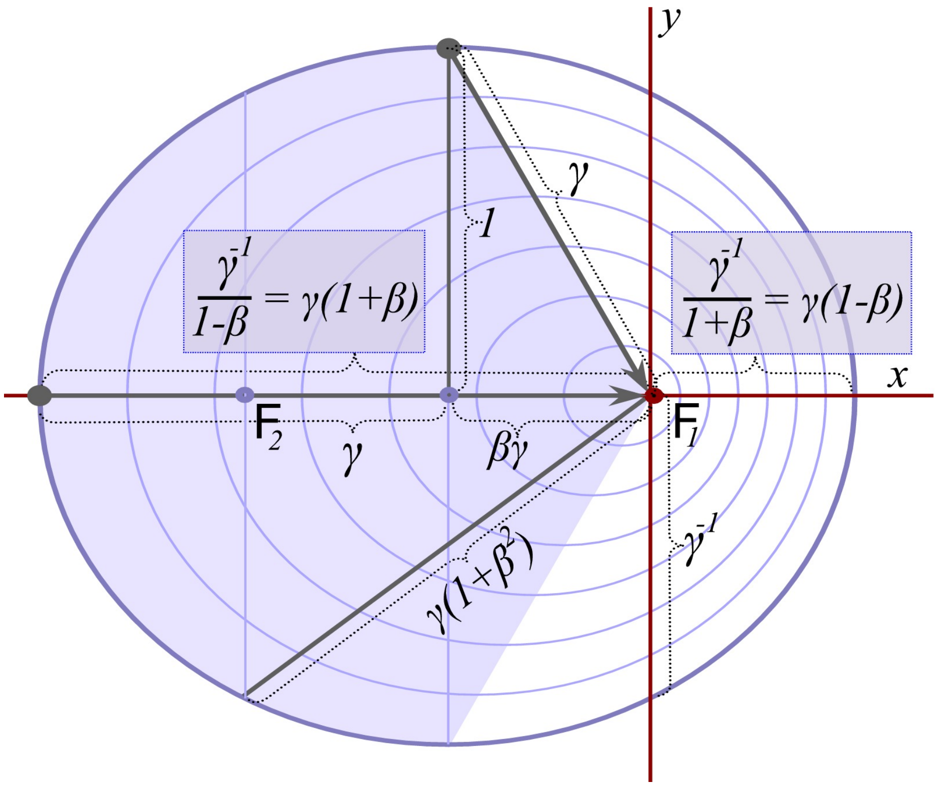

Figure 4.14: Some more properties of the ellipsoid of simultaneity

Figure 4.14 shows some more basic properties which follow from the assumption that the closed curves are ellipses and thus are described by.

$$\left(\frac{x-\beta t}{\gamma\tau}\right)^2+\left(\frac{y}{\tau}\right)^2 = 1 \tag{4.9}$$

Where $\gamma\tau$ is the length of the major axis and τ is the length of the minor axis of the ellipses. The assumption of ellipses is by no means obvious and in section 4.13 we derive that this is true only in case of a Lorentz contraction by exactly a factor of γ.

We turn to figure 4.15 for some further elementary calculations. It does also correspond with the light-cone top view of figure 4.10. The ellipses are the intersections of the x'-planes of simultaneity with the light-cone.

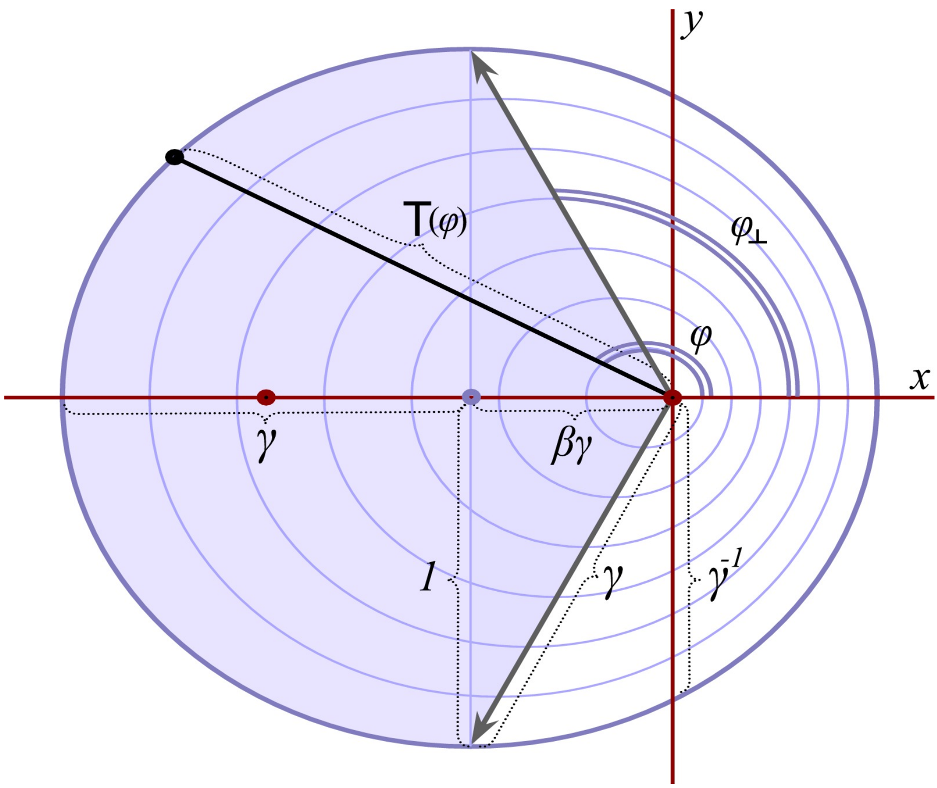

Figure 4.15: Angular sizes for a velocity β

We obtain the parametric expressions of the ellipses by intersecting the lightcone with the planes of constant t'.

$$\text{The lightcone:} \qquad t = -\tfrac{1}{c}\sqrt{x^2+y^2} \tag{4.10}$$

$$\text{The planes:} \qquad t' = \gamma\left(t - \frac{v}{c^2}x\right) \tag{4.11}$$

Setting $c = 1$ as we have done until now and exchanging t' with t at the left hand side of the latter expression so that we can equate the two expressions as $t = f_{cone} = f_{plane}$ and we obtain the second order parametric equation for the ellipses.

$$\text{The ellipses:} \qquad \left(\frac{x - \beta\gamma t'}{\gamma t'}\right)^2 + \left(\frac{y}{t'}\right)^2 = 1 \tag{4.12}$$

The proper time t' of the observer determines the ellipsoid. This means that the ellipsoids represent simultaneous events for the moving observer. The outer ellipse corresponds with proper time $t' = 1$.

The time $\mathsf{T}(\varphi)$ in figure 4.15 denotes the time it takes for light to propagate from the (outer) ellipse to the top of the light cone. We determine the intersection point of the radial line $y = (\tan\varphi)x$ with the (outer) ellipse and obtain for the x and y coordinates and the propagation time T.

$$x = \gamma\left[\frac{1-\beta^2}{1+\beta\cos\varphi}\right]\cos\varphi \qquad y = \gamma\left[\frac{1-\beta^2}{1+\beta\cos\varphi}\right]\sin\varphi \tag{4.13}$$

$$\text{Propagation time:} \qquad \mathsf{T}(\varphi) = \gamma\,\frac{1-\beta^2}{1+\beta\cos\varphi} \tag{4.14}$$

Some special values of T at angles $\varphi = 0$ and $\varphi = \pi$ on the x-axis are.

$$\mathsf{T}(0) = \gamma(1-\beta) \qquad \mathsf{T}(\pi) = \gamma(1+\beta) \tag{4.15}$$

While on the y-axis where $\varphi = \pi/2$ and $\varphi = -\pi/2$ radians we find.

$$\mathsf{T}(\pi/2) = \mathsf{T}(-\pi/2) = \gamma(1-\beta^2) = \gamma^{-1} \tag{4.16}$$

The angle under which the light approaches from the side as seen in the rest frame of the passenger and as visualized in figures 4.11 and 4.12 is found by requiring $\mathsf{T}{=}\gamma$ and is given by.

$$\varphi_\perp = \arccos(-\beta) \tag{4.17}$$

4.10 From ellipsoids to spheres of simultaneity

(The Lorentz transform)

We found that an ellipsoid of simultaneity of a moving observer is an intersection between the light cone and a "plane"of simultaneity like in figure 4.10. Such a ellipsoid of simultaneity is seen as a symmetric *sphere* of simultaneity for the moving observer.

All the events on the ellipsoid reach the moving observer at the same time, and equal objects at these events seem equidistant for the observer. They seem to have equal sizes because their relative viewing angles look equal.

The transformation between the ellipsoid and the sphere of simultaneity is of course the Lorentz transform which will be subject to a more detailed inspection here from this perspective. We can separate the transform into two distinct steps.

1. Transformation from absolute to relative coordinates. (from absolute viewing angles to relative viewing angles), See also section 4.11
2. Corrections due to the Lorentz contraction and Time dilation of the moving observer, See also section 4.12

Figure 4.16 visualizes the Lorentz transform of the spatial component x from ellipsoid to sphere. The x used for the ellipsoid is just the same x as for the rest frame. The events on the ellipsoid do belong to different times but the positions are just the x-positions.

The first step is the transform to the relative positions of the events with regard to the observer. The focus (top of the light cone) continues to move during the time T required for the time it takes for light to propagate from the event to the focus. We have to correct the x coordinate accordingly.

$$x'' = (x - \beta t) \tag{4.18}$$

The second step is a correction by γ because, as we will discuss in section 4.12, a Lorentz contracted observer will observe his environment as stretched by a factor gamma instead.

$$x' = \gamma x'' = \gamma(x - \beta t) \tag{4.19}$$

With this expression we have arrived at the standard form of the Lorentz transform for the spatial coordinate.

Lorentz transform of the spatial components in two steps

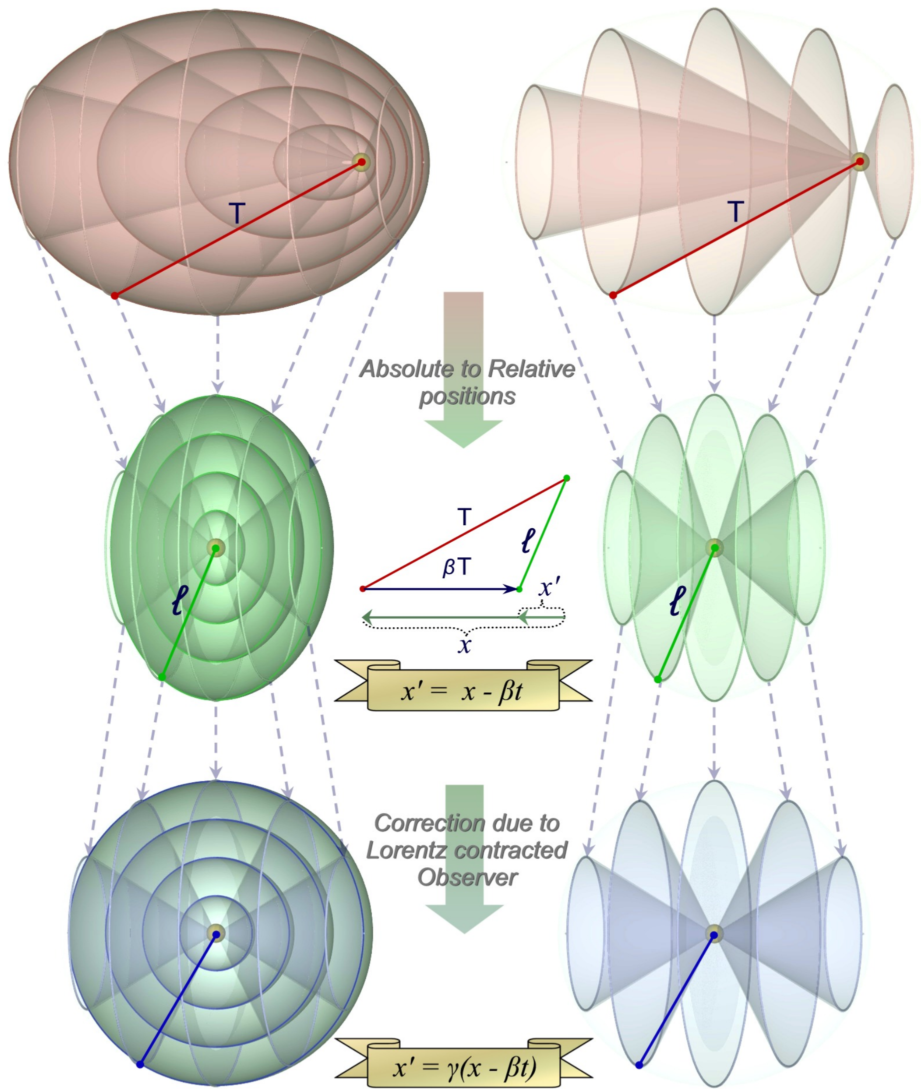

Figure 4.16: Lorentz transform (space components)

Lorentz transform of the time component

Figure 4.17 shows the same two steps as the previous image but now looking from the top and also looking from the side of the light cone. The latter shows us the time axis t of which we will discuss the Lorentz transform.

Our goal is to define the time t' as it is experienced by the moving observer. We first have to redefine simultaneity in correspondence with the assumptions of the moving observer, who considers equidistant events from which light rays reach him at the same time as simultaneously.

We see from figure 4.17 that these planes of simultaneous events are not horizontal but that t' is dependent on x. We have to include this dependency in the transform. The ramp is βx for the ellipsoids of simultaneity, so in this case we could define (just as an example) $t''' = t - \beta x$.

However, our starting point is figure 4.17b which shows us the situation as seen by a moving observer if he would not be Lorentz contracted nor subject to time dilation. It shows the relative positions of the events with regard to the observer at the moment they happened. The ramp depending on the x''-coordinate here is steeper by a factor γ^2 since the relative-position-ellipsoid is shorter by a factor γ^2. This gives us.

$$\begin{aligned} t'' &= t - \beta\gamma^2\, x'' \\ &= t - \beta\gamma^2(x - \beta t) \\ &= \gamma^2\big(\, t - \beta x\,\big) \end{aligned} \tag{4.20}$$

Going from figure 4.17b to 4.17c we need to compensate for the two effects which modify the observations of the moving observer. We have to increase the spatial dimension in the x-direction by γ because this direction appears stretched for the Lorentz contracted observer. Next we have to decrease the time by a factor γ because of the time dilation which the moving observer experiences.

$$t' \;=\; t''/\gamma \;=\; \gamma\,(t - \beta x) \tag{4.21}$$

We have recovered the standard Lorentz transform. We have gone from the ellipsoids of simultaneity for the moving observer as seen in the rest frame, to the symmetric spheres of simultaneity as experienced by the observer.

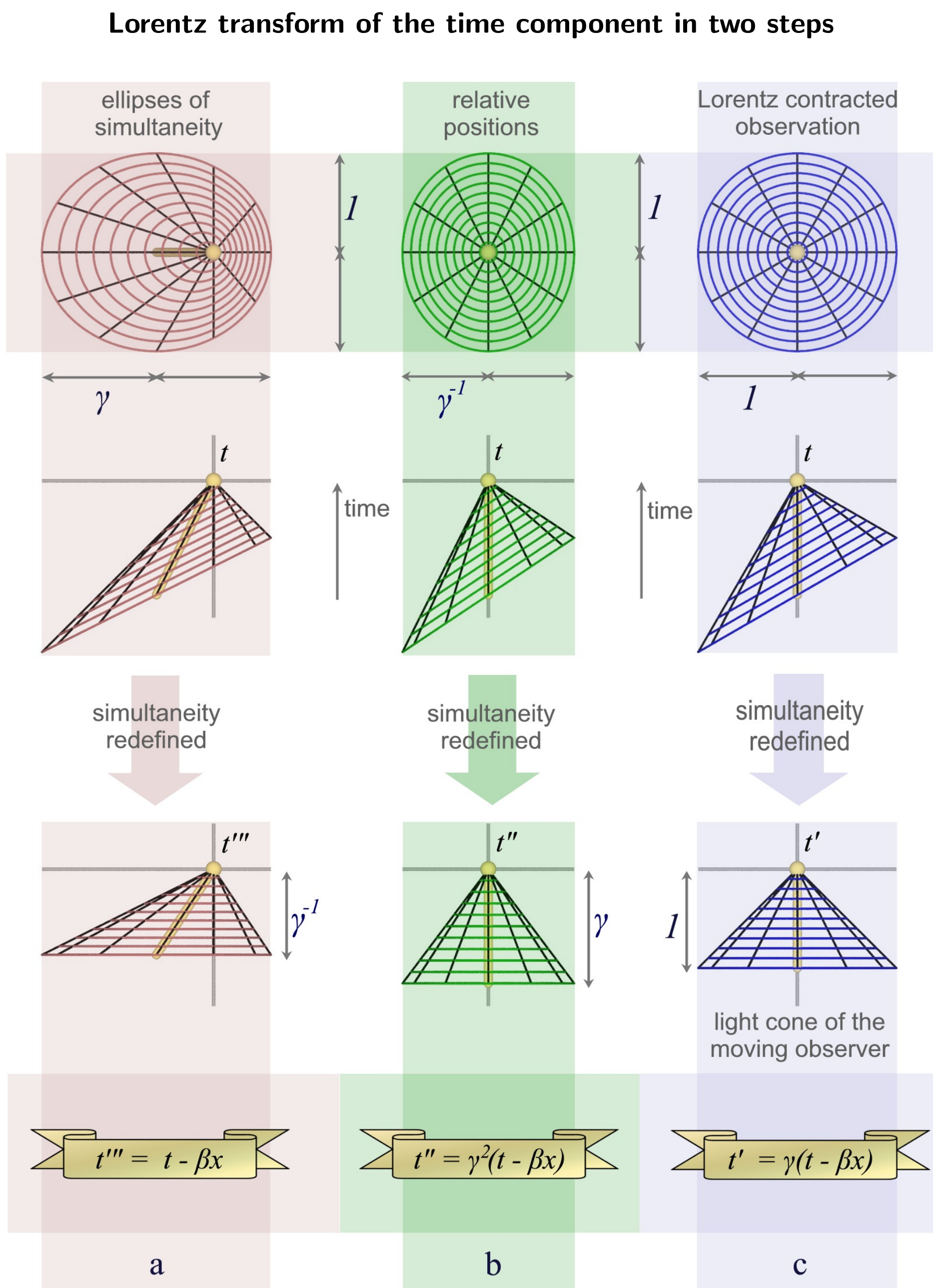

Figure 4.17: Lorentz transform (time component)

4.11 Step 1: From absolute to relative positions

(From absolute viewing angles to relative viewing angles)

Figure 4.18 shows the ellipsoids of simultaneity with absolute positions (left) and relative positions (right) with regards to the moving focal point. Intermediate positions 1 through 5 of the moving focal point correspond with the moments at which the events occur on the ellipsoid.

We obtain the relative positions if we hold the position of the focal point fixed. The ellipsoid with absolute positions is stretched by γ while the ellipsoid with relative positions is Lorentz contracted by γ.

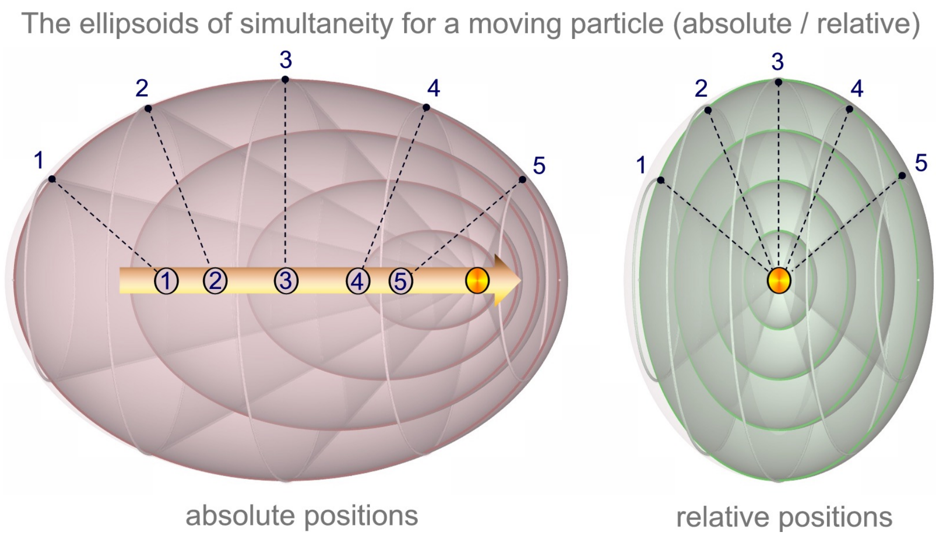

Figure 4.18: The ellipsoids of simultaneity with absolute/relative positions

We can understand the Lorentz contracted ellipsoid of simultaneity at the right side as follows: Imagine filming a moving sphere which is Lorentz contracted accordingly. We keep the center fixed. Next we start freezing vertical stripes of the image, starting at the left and going to the right.

By freezing the stripes we capture events on the ellipsoid at different times. The size and shape of the resulting ellipsoid is the same but what we have done is introducing a different simultaneity: The simultaneity of the moving observer in his rest frame.

This part of the Lorentz transform is just a simple Euclidian transformation as shown in figure 4.19. The x-coordinate is shifted with the velocity β.

$$x'' = (x - \beta t) \tag{4.22}$$

The transformation to relative positions also gives us the relative viewing angles. The trajectory ℓ in figure 4.19 shows the absolute viewing angle while the trajectory ℓ'' shows the relative viewing angle.

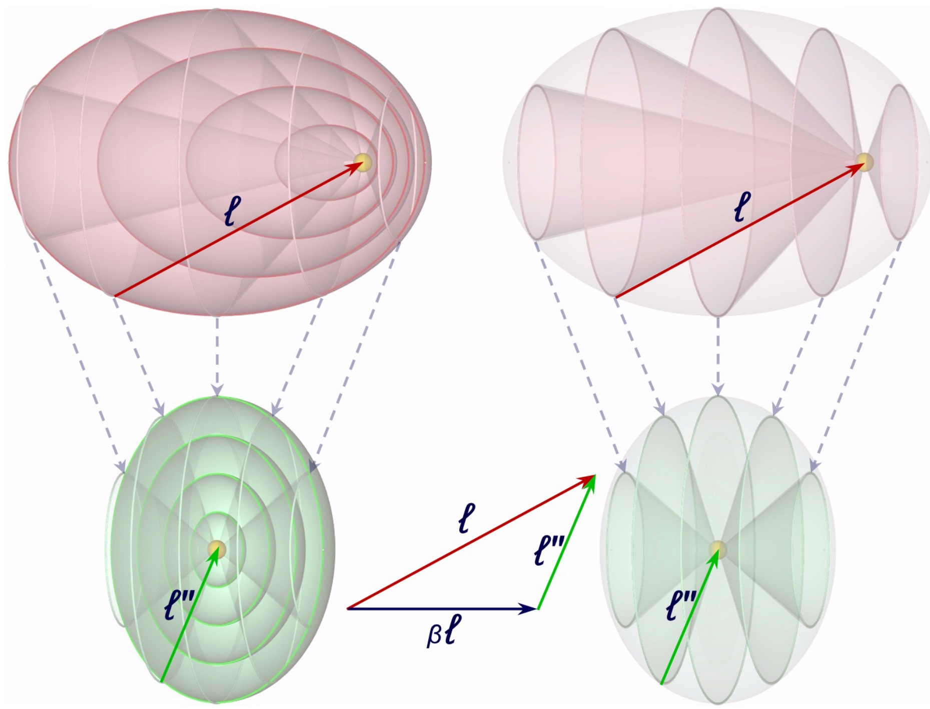

Figure 4.19: From absolute positions (angles) to relative positions (angles)

The two trajectories $\vec{\ell}$ and $\vec{\ell}''$ are thus related by a simple vector addition. The relative viewing angle is the angle as seen by a non-Lorentz compressed observer. The relative viewing angles are not yet spherically symmetric as they should be for the equidistance requirement. The second part of the Lorentz transform which compensates for the Lorentz compressed state of the observer will give rise to a spherically symmetric result.

4.12 Step 2: Viewing while Lorentz contracted

Here we ask ourself how we would see our world if everything is scaled in one particular direction by a certain factor. Would we notice this or would we simply not see this? It's much easer to be convinced that we would not notice a scaling in all directions with the same factor, but what mechanism could prevent us to see such a unidirectional scaling as required for Lorentz invariance?

Figure (4.20) is intended to visualize this. It shows our passengers Lorentz contracted under various angles relative to the velocity. We use again the *camera obscura* as the simplest example of a camera and this time we use an over-sized version for clarity. For a moment we won't bother with the propagation times of the light ray for a first conclusion.

Technically, that what we *see* is equal under different conditions if the same rays always hit the same receptors of the eye (or the image sensor). The same nerve cells in the brain are activated (and the same file is constructed in a digital camera). We see in figure (4.20) that this indeed seems to be the case for our camera obscura.

For the next step we need to consider the time delays in ray propagation. Figure (4.20) has recorded events over the time it takes the light rays to move from the passengers, via the focus, to the screen of the camera obscura. All rays are at the focus points at the same time. A focus point is the top of the light-cone as in figure (4.17b) and it's also the bottom of the light cone of the rays going to the screen.

Figure (4.20) shows a Lorentz compressed image because all events are recorded *relative* to the position of the moving at the time the event occurs. We translate along with the moving focus during the recording keeping it fixed. Since the positions are Lorentz contracted at each moment we add an event we get an image representing the entire recording which is Lorentz compressed as well.

Conclusion:
An observer at the focus would assume that all events are simultaneous because, firstly: the light rays reach him simultaneously, and secondly: he assumes that the events are equidistant because the size of our model passenger is the same independent of the direction. (see section 4.3). The moving observer concludes that his notion of simultaneity is different.

Lorentz contraction goes unnoticed with a lorentz contracted camera.

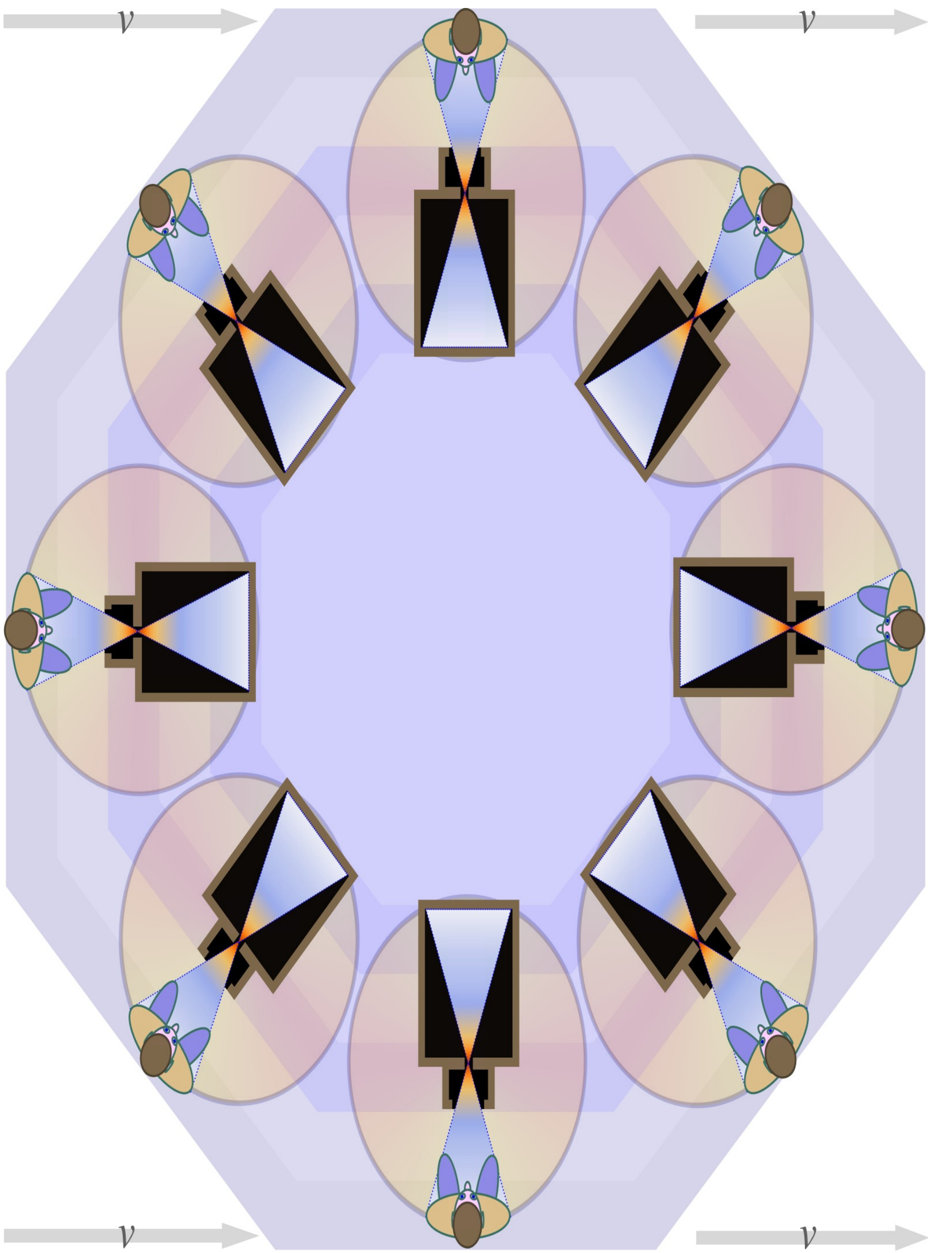

Figure 4.20: Passengers Lorentz contracted in various directions

4.13 Simultaneity from the Spherical Mirror clock

An elementary example which combines all the phenomena we have encountered sofar is that of the Spherical Mirror clock as shown in figure 4.21. A light flash at the center of the sphere at t=0 will be reflected by the inner surface and will be focussed back to the center after a time proportional to the radius. A series of impulses at a constant rate is the result and we have in principle an elementary clock.

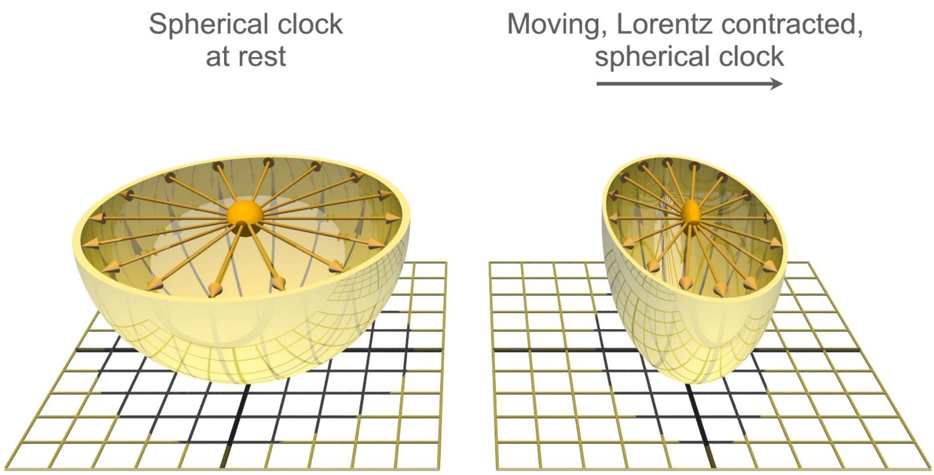

Figure 4.21: Sphere clock at rest and moving with 0.8c

We now ask ourself what it takes to achieve the same periodically refocussing with a moving setup. We'll show that the way to achieve this in classical physics leads us to all the effects of special relativity. Lorentz contraction, Time dilation and non-simultaneity.

We will see that the sphere should be an ellipsoid contracted by γ in the direction of motion. The factor γ is the only factor which results in a refocussing of the rays in a (moved) focal point. Any other factor of contraction or stretch doesn't work, and will not refocus the rays.

The upper part of figure 4.22 shows us the top-view of the moving ellipsoid at a sequence of time steps plus the paths of the rays reflecting at the moving surface. The way this geometrically works becomes clear if we show the mirror surface at the points where the rays hit the surface, as is shown in the lower half of figure 4.22.

The resulting object is another ellipse, which is now stretched (instead of contracted) by a factor γ. The shape has to be an ellipse because of the characteristic two focal points of ellipses plus the rule that any path with one reflection between the two focal points is equally long.

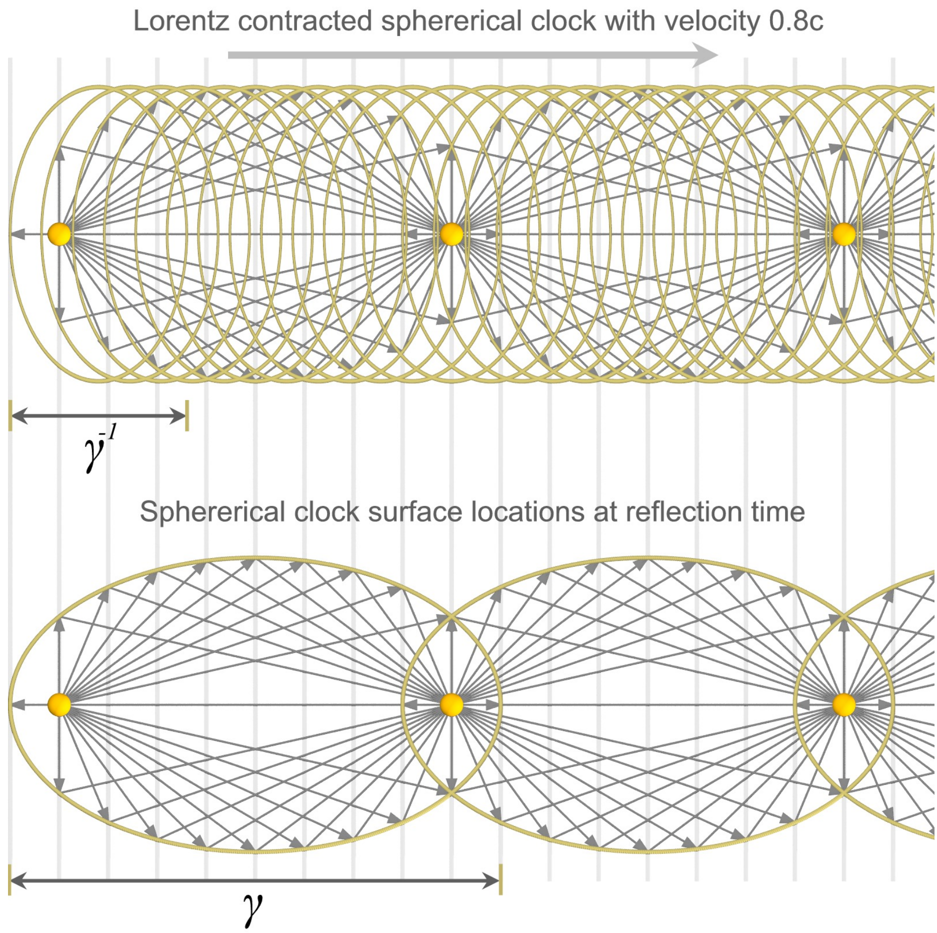

Figure 4.22: Moving (Lorentz contracted) sphere clock at 0.8c

These two properties of ellipses guarantee that the rays are refocussed, not only at the same point but at the same time as well. The stretch factor γ causes a slowdown of the tick rate of our elementary clock: We have recovered the time dilation factor γ.

We also see that the rays are reflected at different times instead of all at once as in the case in which our clock is at rest. From figure 4.23 we see that the reflections take place on planes of equal x'. The previous image, figure 4.22, is also the top view of this image.

The rays come together at the focal points at the same time. An observer at a focal point sees all the reflections happening at the same time corresponding with the plane of simultaneity in the moving reference frame.

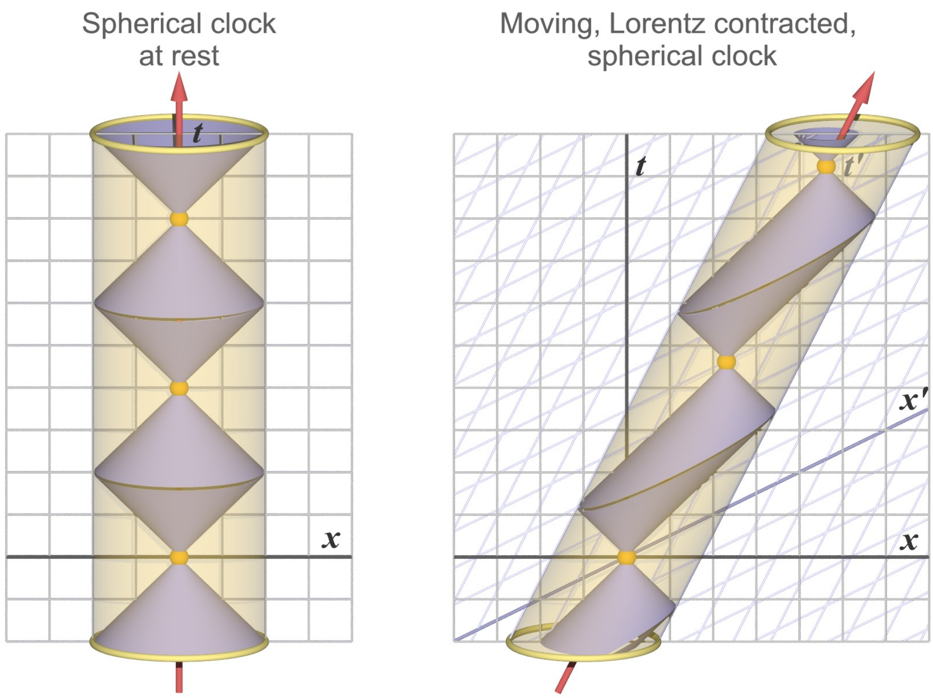

Figure 4.23: Minkowski diagrams: Sphere clock at rest and moving at 0.5c

The angles of reflection are determined by the surface normals of the stretched ellipse because it is the surface of this ellipse which connects all the reflection events.

The Lorentz contracted spherical clock only works because it is contracted by exactly the factor γ. Any other factor of contraction does not work. The factor gamma is the only factor that leads to an elliptic intersection between the light cone and a plane as shown below in figure (4.24).

Any other factor of contraction leads to a non-elliptic curve. If we start with a spherical clock and contract it by $'a'$ instead of γ as in figure (4.24) then the relation between x and y is given by the quadratic expression.

$$a^2x^2 \ + \ y^2 \ = \ 1 \tag{4.23}$$

When the center C of the Lorentz contracted ellipse is at focus point F_1, then the light ray starts in the direction of reflection point $p(x,y)$. Figure (4.24) shows the location of the center C at the moment of reflection. The center of the ellipse then continues to move with a velocity β during the time t which the reflected ray needs to propagate to the focus point F_2.

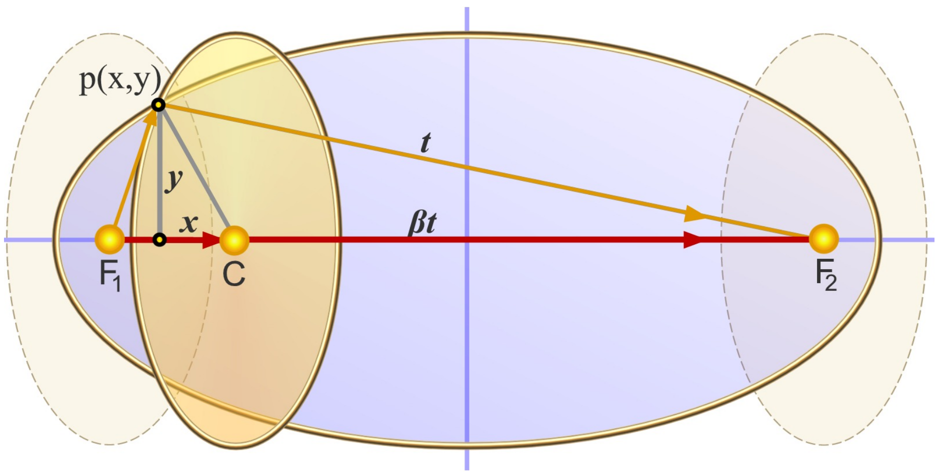

Figure 4.24: Direct calculation of the lightcone ellipse (v=0.866c)

The time t is obtained by solving the resulting equation in t and given by.

$$t \ = \ \beta\gamma^2 x - \gamma\sqrt{1 \ + \ (\gamma^2 - a^2)\, x^2} \tag{4.24}$$

$$x_f \ = \ \gamma^2 x - \beta\gamma\sqrt{1 \ + \ (\gamma^2 - a^2)\, x^2} \tag{4.25}$$

With t known we can calculate the coordinate $x_f = x + \beta t$ of the reflection point $p(x,y)$ relative to the focus F_2. We do find the expected stretch by a factor of γ^2 but also an unwanted term with a non-linear dependency on x which is canceled only if the Lorentz contraction factor 'a' is equal to γ.

4.14 Reversal of Lorentz contraction

Understanding the mechanism of how and why we perceive different simultaneities in different reference frames has been the primary purpose of this introductory chapter. A short review is now given of how, as a result, the physical effect of Lorentz transform is reversed when going from one to another reference frame.

Lorentz contraction is a physical effect, it is a property of the stable, moving solutions of the elementary wave-equations for both electro-magnetism and particles with mass. Lorentz contraction is a physical effect on it's own, next to time dilation and simultaneity. It is not caused by non-simultaneity. It is the reversal of the effect which is caused by non-simultaneity.

Figure 4.25 shows all there to know. The upper half shows a moving train with a proper length of L which is physically Lorentz contracted to L/γ. It shows the trajectories t' of the moving train and how it intersects with the plane of simultaneity x' of the moving observer.

The begin-to-end length of this intersection is stretched by a factor γ^2 to a length of γL on the x-axis. (This factor $\gamma^2 = 1 + \beta^2 + \beta^4 + \beta^6$... can be checked with a simple geometric trick which is also shown in the image)

This length of γL is first scaled down again by a factor γ^2 when we perform the Euclidian transformation $x'' = x - \beta t$ as shown in section 4.11, and secondly, when we take the effect of the Lorentz contraction on the observation into account, it is stretched by a factor γ in the direction of motion, as shown in section 4.12.

This leads to an effective length of L as seen by the moving Lorentz contracted observer. The moving observer sees himself and the length of the train in which he travels in a way which is independent of his velocity as required by the invariance of physics under boosts.

The lower part of figure 4.25 shows the reversed case. The train at rest has a length L, a factor γ longer as the moving train.

The intersection of the trajectories t of the train at rest with the plane of simultaneity x' is L as well on the x-axis. This now is a factor of γ shorter as the intersection of the moving train which we found to be γL. This means that the Lorentz contraction is reversed in the moving frame.

Reversal of the Lorentz contraction demonstrated with trains

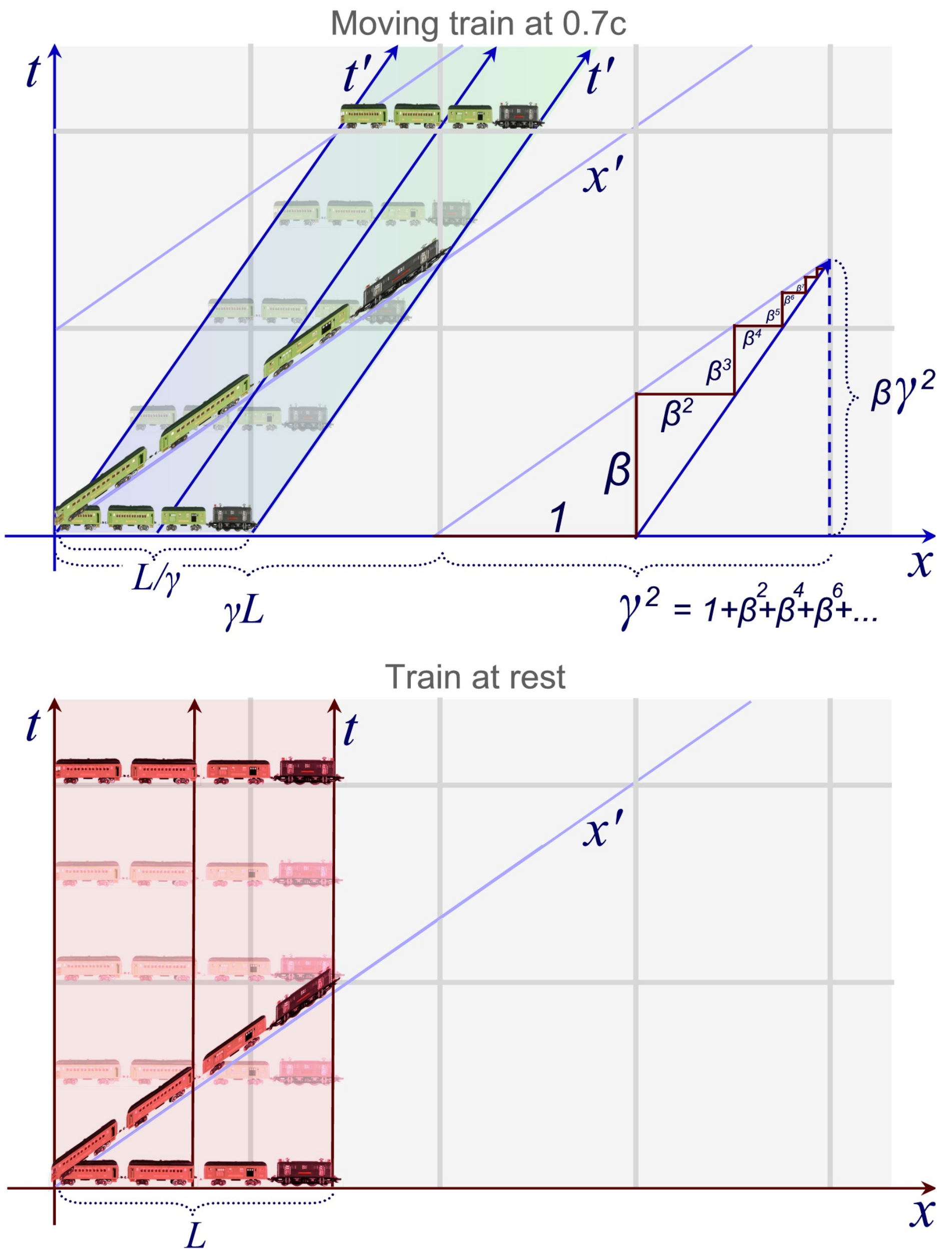

Figure 4.25: Reversal of the Lorentz contraction

4.15 Reversal of Time dilation

Time dilation is, like Lorentz contraction, a physical effect. It is caused by the longer signal paths in the stable, moving solutions of the elementary wave equations of electro-magnetism and matter-waves.

We have seen from our bouncing photon clock examples that Time dilation *requires* the effect of Lorentz contraction in order to be *isotropic* (independent of the direction in which photon bounces). It assures that the time it takes for the photon to bounce back and forward is the same in all directions.

It would not be possible to define a unique time dilation factor without such an isotropic definition. Only a Lorentz contraction by exactly a factor of γ allows us to formulate a unique definition of time dilation.

Time dilation is reversed by non-simultaneity via a redefinition of the plane of simultaneity x'. A redefinition of a coordinate system is in principle "non-physical". Distant events are related in different ways depending on coordinate systems chosen consistently in such a way that it is most natural for observers which are at rest in the associated reference frames.

Time dilation becomes a physical reality if we can compare events (with different histories) at the same time and the same place. The standard example here is the twin "paradox". We'll use figure 4.26 to first discuss the two alternatives in a single reference frame.

The left hand side image shows how the aging of the moving twin brother has slowed down by a factor *gamma*. This because he traveled away first with a velocity $+\beta$ and returned halfway with a velocity $-\beta$. The longer signal paths within the moving observer slowed down his physical processes by a factor γ.

The right hand side image shows the reversed case. It starts the same but now the moving twin brother continues to move away and his twin brother at rest decides to chase him instead so that they will meet again after the same total time. (in the rest-frame).

The initially resting twin brother starts his chase at the middle of his journey, at half-time. That is: the proper time t''_2 spend at the second half of his voyage is equal to the time t_1 spend in the first half.

Reversal of the Time dilation demonstrated with the twin voyages

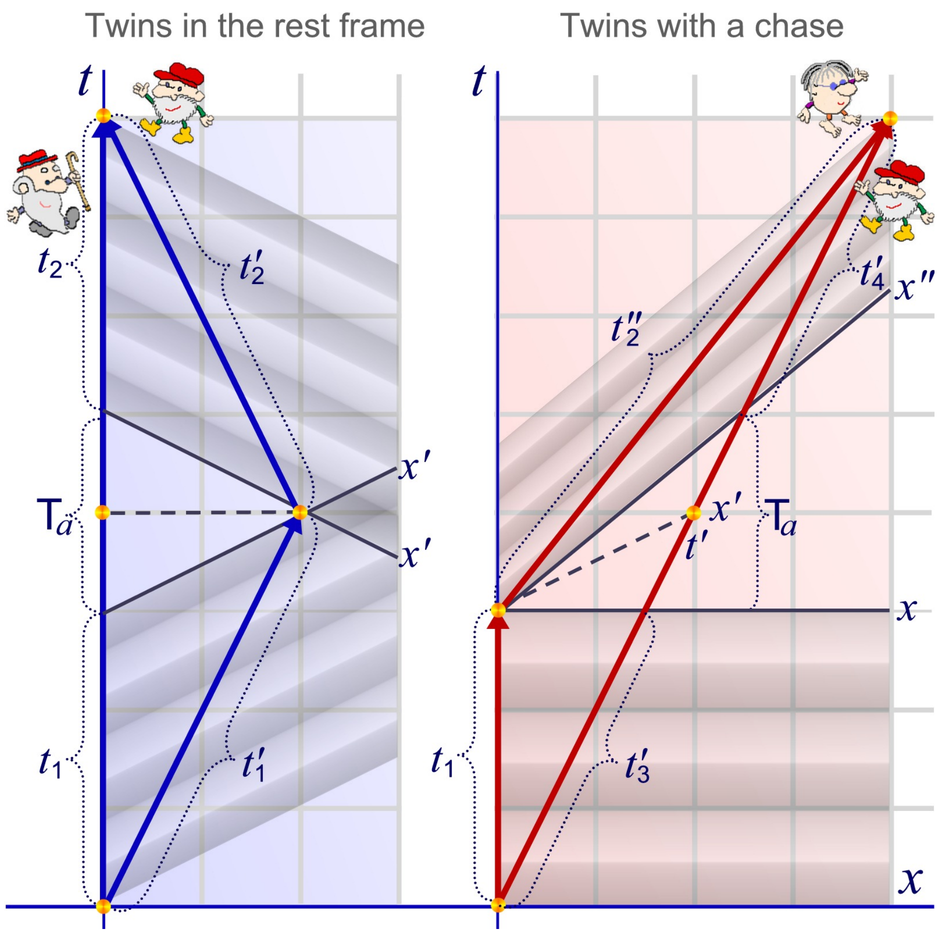

Figure 4.26: Reversal of the Time dilation

The physical time dilation during the second part of his voyage is such that the total proper time of the path $t''_2 + t_1$ is less by a factor γ^2 due to this higher time dilation. This is a factor γ less as his twin brother who has traveled with continues velocity $+\beta$ during his entire voyage[2]

The twin brother who accelerated halfway to catch up with his brother has aged less by a factor γ compared with his non-accelerating twin-brother.

[2] The mathematics of this was handled in the previous chapter, in section 3.3

For sofar we have only looked at a single reference frame, but we have nevertheless seen how the non-accelerating twin brother ages less by a time dilation factor γ. This factor is entirely explained physically by the longer path-lengths of signals within moving observers.

During all this we never needed to refer to non-simultaneity and non-simultaneity is as such not required to explain the two cases. It did show up indirectly however. The half-time points of the two travelers in the right-side image are not simultaneous in the rest-frame but they are simultaneous in the moving frame instead, they are on the same x'-axis.

Simultaneity and the reversal of time dilation

The reversion, required by he invariance of physics under boosts, says that *both* brothers, at any moment of time, can say that his twin brother is aging slower as himself by a factor γ. This is possible only by a redefinition of simultaneity, a redefinition in such a way that their notion of simultaneity corresponds most naturally to what they observe.

Basically, an observer will assume that two equidistant events are simultaneous if the light-rays of both events reach him at the same time and he judges equidistance by the observation that identical objects at the events have equal viewing angles.

Figure 4.26 shows, at the left side, the two different reference frames of the accelerating observer. In the first of these frames the twin brother at rest ages by a time t_1. The acceleration halfway then skips a time T_a after which the passenger at rest ages by time t_2. Now the total time $t_1 + t_2$ is a factor γ less as the proper time spend by the moving observer. The moving observer may claim that his twin-brother at rest was aging slower by a factor γ.

So both twin brothers can claim that the other is aging less by a factor γ. Both are correct in the coordinate systems they use. The difference is the acceleration and the time T_a which is skipped. The twin brother at rest ages with the total time $t_1 + \mathsf{T}_a + t_2$ which is a factor γ *higher* as the proper time by which the moving twin brother aged.

So we have the following proper traveling times for the left hand side of figure 4.26:

$$t'_1 + t'_2 \;=\; \gamma(\; t_1 + t_2 \;) \tag{4.26}$$

$$t_1 + \mathsf{T}_a + t_2 \;=\; \gamma(\; t'_1 + t'_2 \;) \tag{4.27}$$

A redefinition of the coordinate system is in principle a non-physical operation. It merely adapts, reorders, the description of the environment in a way which is most natural to the moving observer.

One can for instance say that the observer at rest "ages" by a time T_a during the acceleration of the traveling observer. This is not incorrect but it is non-physical because we are only dealing with a redefinition of the reference frame during the acceleration.

Figure 4.26 shows the same effect on the right hand side for the case of the twin brother at rest who decides to chase his non-returning traveling twin brother. The twin brother which accelerates has again basically two different reference frames during the two different stages of his voyage.

The chasing twin-brother can claim during the first stage that his twin brother ages less since $t_1 = \gamma t'_3$. He can also claim this during the second stage of his voyage because $t''_2 = \gamma t'_4$. The acceleration at half-time again skips a time T_a due to the redefinition of his coordinate system. This skipped time T_a corresponds with a skipped proper time of $\mathsf{T}'_a = T_a/\gamma$ in the reference frame of his non-accelerating brother.

This gives us the following relations for the proper traveling times on the right hand side of figure 4.26:

$$t_1 + t''_2 \;=\; \gamma(\; t'_3 + t'_4 \;) \tag{4.28}$$

$$t'_3 + \mathsf{T}'_a + t'_4 \;=\; \gamma(\; t_1 + t''_2 \;) \tag{4.29}$$

Again, both brother may claim that the other is aging less by a factor γ, and they may claim so due to the redefinition of their coordinate systems.

4.16 Simultaneity from Huygens principle

The Huygens principle, in its most elementary form, states that the light wave front is always in the direction of the propagation. All but few effects in optical geometry can be explained via this simple statement.

Huygens principle plays an equally important role in matter waves, again the propagation direction and the wavefront direction are always aligned.

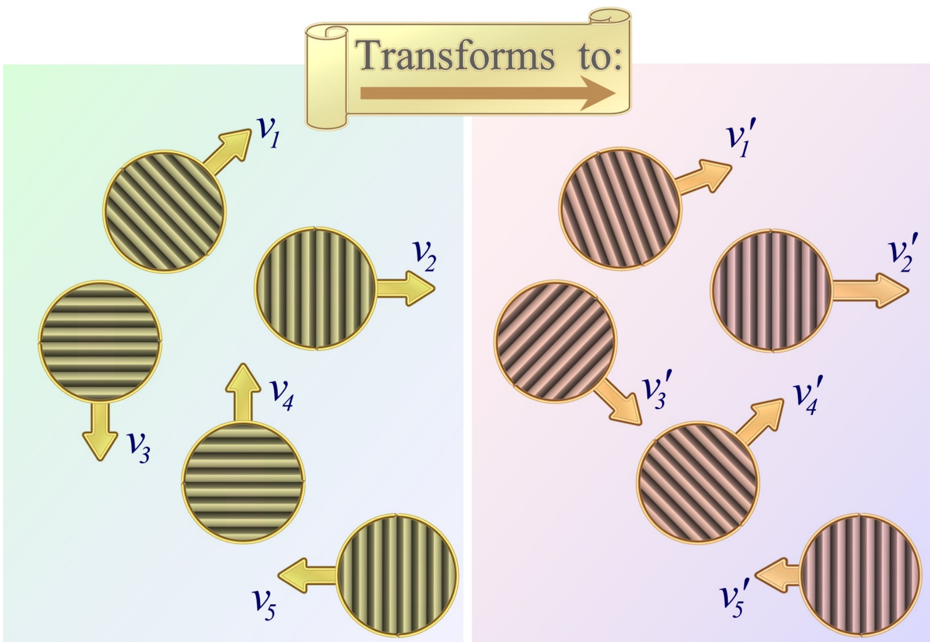

Figure 4.27: Huygens principle

Interference plays an elementary role in the physics of propagation and the wave front represents the lines along which the phase is added maximally or minimally.

If physics is to remain invariant under boosts then we should expect that Huygens principle is equally well valid in all other reference frames. This now leads to the requirement of non-simultaneity. As the vectors of motion change in direction going from one reference frame to another, so should the wavefront direction change.

4.17 Simultaneity and the light wavefront direction

The requirement that the wavefront of electromagnetic radiation is always oriented in the same direction as its velocity requires the introduction of non-simultaneity. The standard example of the bouncing photon clock shown left in figure 4.28 is Lorentz transformed to the middle image showing the wavefront rotated to the transformed direction of propagation. Such a rotation can only be obtained if t' depends on the x-coordinate.

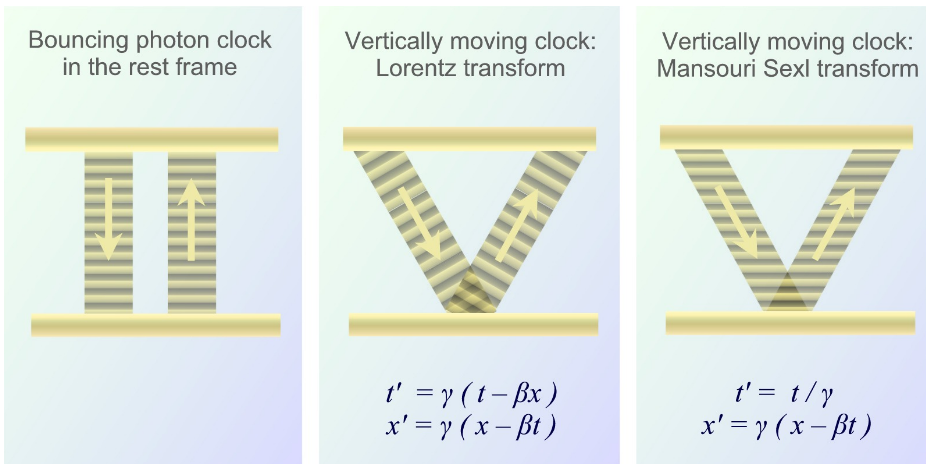

Figure 4.28: The bouncing photon clock moving vertically

An alternative transformation which lacks the non-simultaneity is the Mansouri Sexl transform shown at the right. The Mansouri Sexl transform's expression for t' is the proper time of moving observer. The proper time is obtained by setting x' to zero after first substituting x with the inverse Lorentz transform.

$$t' \quad = \quad \gamma(t - \beta x) \quad = \quad \gamma(t - \beta\gamma(x' + \beta t')) \tag{4.30}$$

$$x' \equiv 0 \quad \Longrightarrow \quad (1 + \beta^2\gamma^2)t' = \gamma t \quad \Longrightarrow \quad t' = t/\gamma \tag{4.31}$$

The Mansouri Sexl transformation introduces Lorentz contraction and time dilation as required by special relativity. However, the missing ingredient in this transformation is non-simultaneity. No wave-front rotation occurs.

The wavefront rotation of light due to non-simultaneity

We want to have a closer look at the mechanism by which non-simultaneity leads to a rotation of the light wave-front. Figure 4.29 shows the vertical timing bands with the Δt resulting from non-simultaneity. The phase of wavefront has propagated further in bands positive Δt and it has propagated less in bands with negative Δt. The result as we can see is a rotation of the light wave front.

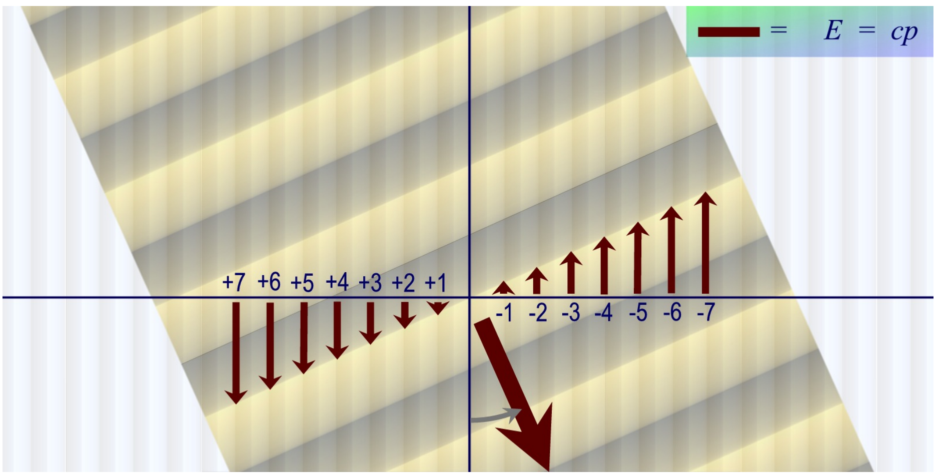

Figure 4.29: Wave front rotation from non-simultaneity (for light)

Any rotation operation can always be decomposed into two orthogonal skew operations, which can be symbolically expressed as.

$$\circlearrowleft \quad = \quad \leftrightarrows \quad + \quad \downarrow\uparrow$$

$$\textit{Wavefront Rotation} \quad = \quad \textit{Mansouri Sexl} \quad + \quad \textit{Non Simultaneity}$$

The first skew operation occurs with the Mansouri Sexl transform as shown in figure 4.28. The second skew operation is a result of non simultaneity as it is shown above in figure 4.29. Together they rotate the wavefront to the new direction of motion in the new reference frame.

Let's now derive figure 4.29 from a vertically propagating light wave front via the standard Lorentz transform. The expression which gives us the phase of the initial wavefront is given by. (Using velocity $v = \{0, -c, 0\}$)

$$\psi(x^\mu) \quad = \quad \exp\left\{ \tfrac{iE}{\hbar}\Big(-t \; - \; y \;\Big)\right\} \tag{4.32}$$

We operate on this expression with a Lorentz transform which corresponds with a reference frame moving with a speed of $-v$ in the x-direction so that our light beam should gain a velocity component v in the x-direction.

We want to express equation (4.32) in terms of the new coordinates t', x' and y'. In order to do so we need the *inverse* Lorentz transform of the coordinates to pick up the value of the phase in the original reference frame.

$$x'^\mu \quad = \quad \Lambda^{-1} x^\mu \tag{4.33}$$

The individual components t', x', y' and z' of the inverse Lorentz transform are given by.

$$t' = \gamma\big(\; t + (-v)x \;\big), \qquad x' = \gamma\big(\; x + (-v)t \;\big), \qquad y' = y, \qquad z' = z \tag{4.34}$$

Which then gives us for the phase of the transformed wave-function.

$$\psi' \quad = \quad \psi(\Lambda^{-1} x^\mu) \tag{4.35}$$

$$\psi' \quad = \quad \exp\left\{ \tfrac{iE'}{\hbar}\Big(-t' \; + \; \beta\, x' \; - \; \sqrt{1-\beta^2}\; y' \;\Big)\right\} \tag{4.36}$$

Where $E' = \gamma E$, The energy of the beam is higher in the new reference frame. The expression shows that our eigenfunction has indeed gained a velocity component v in the x-direction. The total velocity in the new reference frame is still c since.

$$\Big(\beta\Big)^2 \; + \; \Big(\sqrt{1-\beta^2}\Big)^2 \quad = \quad 1 \tag{4.37}$$

The velocity c of light is reference frame independent under Lorentz Transform as it should be for the invariance of physics under boosts.

4.18 The wavefront rotation of matter waves

The rotation of the light wave front due to a difference in simultaneity corresponds with objects which have a velocity of c. The direction of the motion is arbitrary but the velocity is fixed. Matter waves of particles with mass can propagate with any velocity between zero and c. We expect that the direction of the wave front corresponds with the direction of the velocity of the particle just like in the case of electromagnetic radiation.

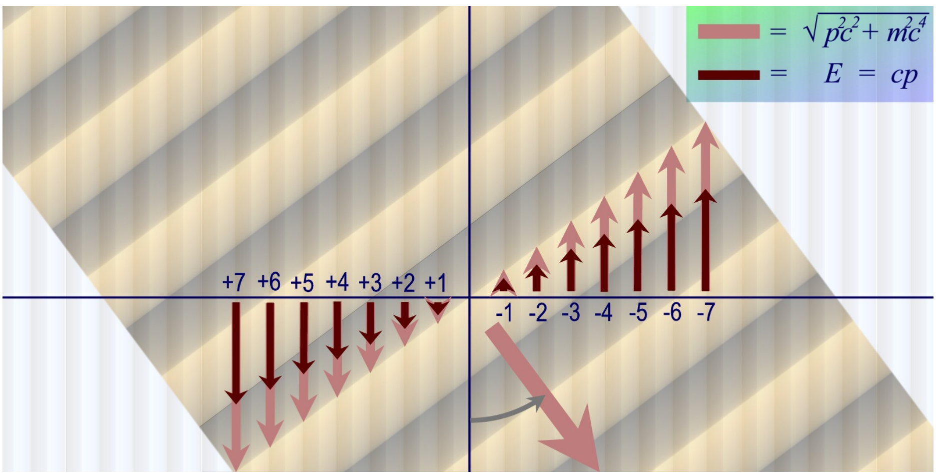

Figure 4.30: Wave front rotation from non-simultaneity (with mass m)

The change in direction is larger for slower moving particles when we go from one reference frame to another. We expect that the phase change due to a different simultaneity needs to be larger to obtain a larger rotation of the wave front. This is indeed the case and shown in figure 4.30

$$\frac{\partial^2\psi}{\partial t^2} - c^2\frac{\partial^2\psi}{\partial x^2} - c^2\frac{\partial^2\psi}{\partial y^2} - c^2\frac{\partial^2\psi}{\partial z^2} \;=\; m^2c^4 \tag{4.38}$$

The extra phase shift stems from the addition of a mass term m^2c^4 in the wave equation which gives us the Klein Gordon equation. The eigen functions of this equation can be written in the same way as those of the electromagnetic field,

$$\psi \;=\; \exp\left\{\tfrac{i}{\hbar}\big(-Et \;+\; p^x x \;+\; p^y y \;+\; p^z z\;\big)\right\} \tag{4.39}$$

but with an additional mass term leading to the classical relativistic energy-momentum relation for a particle with mass m.

$$E^2 \ = \ p_x^2c^2 \ + \ p_y^2c^2 \ + \ p_z^2c^2 \ + \ m^2c^4 \tag{4.40}$$

The mass term m is Lorentz invariant, it is the same in all reference frames. We can express ψ explicitly in terms of the velocity as follows.

$$\psi \ = \ \exp\left\{\tfrac{iE}{\hbar c}\big(- ct \ + \ v^x x \ + \ v^y y \ + \ v^z z \ \big)\right\} \tag{4.41}$$

To obtain figure 4.30 we start with a particle with mass moving in the y-direction with a speed v^y.

$$\psi \ = \ \exp\left\{\tfrac{iE}{\hbar c}\big(- ct \ + \ v^y y \ \big)\right\} \tag{4.42}$$

We perform a Lorentz transform corresponding with a speed of $-v^x$ so that we expect our particle to gain a velocity component v^x in the x-direction.

$$t = \gamma\big(\ t' + (-v^x)x' \ \big), \qquad \text{and} \qquad y = y' \tag{4.43}$$

We assume lower speeds so that we can approximate $\gamma \approx 1$ and we obtain.

$$\psi \ = \ \exp\left\{\tfrac{iE}{\hbar c}\big(- ct' \ + \ v^x x' \ + \ v^y y' \ \big)\right\} \tag{4.44}$$

This showns that our eigenfunction has indeed gained a velocity component v^x in the x-direction. For higher speeds it's slightly less transparent as a result of the relativistic speed addition rules. Note that the effect of wave-front rotation is essential at any speed, even at very low speeds.

The particles which build up our direct environment move at slow "non-relativistic" speeds. Between quotes because it's the relativistic effect of non-simultaneity which rotates the wave-fronts of the matter-waves so that they always point in the direction of the particle's relative speed, the speed relative to our own speed.

4.19 Negative energy waves and wavefront rotation

It is custom to associate anti-particles with negative values of E in expression (4.39). Now what happens in this case with the direction of the wave front? It rotates the other way around! This does not correspond with what we expect from the transformed motion of the particle, however, it's correct when we assume that the particle was actually moving in the opposite direction.

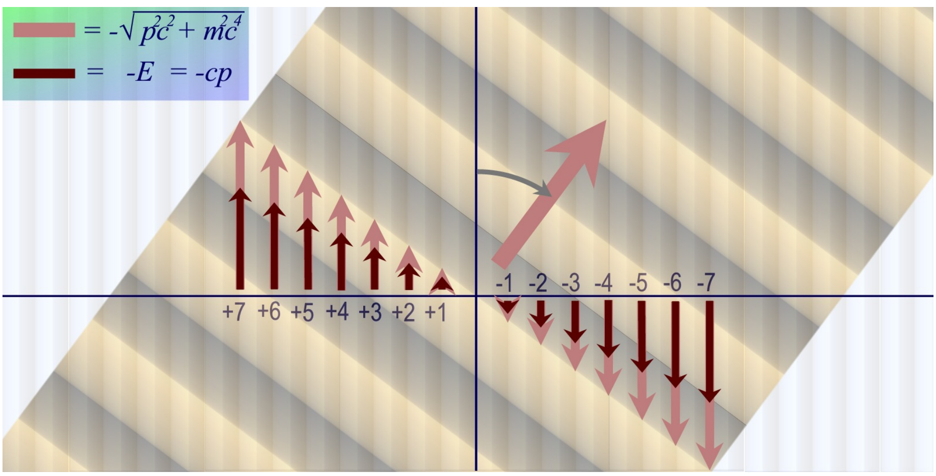

Figure 4.31: Negative energy Wave front rotation.

This is shown in figure 4.31. We could expect this since the wave-front actually propagate to the other direction because of the sign changes. We get further confirmation if we construct a localized wave-packet with an (average) energy-momentum. The localized wave-packet as a whole does also propagate in the opposite direction.

It has been often stated in the Wheeler-Feynman tradition that negative energy anti-particles are moving "backward in time" which would correspond with the reversion of the direction. Such a time-symmetry for elementary particles does not look totally unreasonable. Today however, when for instance anti-hydrogen is routinely produces with modern equipment, we do not really see a reversal of the arrow of time for anti-matter in terms of causality or the second law of thermodynamics.